R.7 Further Topics in Factoring

In this section we need to take an expanded look at factoring. We learned how to do all of the core factoring techniques in chapter 4 and we have used factoring extensively since then.

However, we need to “upgrade” our factoring skills to include a few techniques that turn out to be useful for those who wish to take a Calculus course in the future.

Lets begin by doing an example that starts with factoring binomial GCF’s as we learned before, and quickly build from there.

Example 1:

Factor.

a. $6b^2(b+2)-7(b+2)$
b. $4x^3-4x-x^2+1$
c. $x(3x-1)^3-(3x-1)^2$
d. $x(x+2)^4+2(x+2)^3$

Solution:

a. First, we should notice that the polynomial has a common binomial factor of *b + 2*. So we can simply factor it out as a GCF from each of the terms as follows

$$\begin{aligned}&6b^2(b+2)-7(b+2)\\=\ &(b+2)(6b^2-7)\end{aligned}$$

Since $6b^2-7$ cannot be factored any further, we are done.

b. In this example, we need to start by factoring by grouping in order to see if we have a common binomial factor. The first two terms clearly have a 4x in common and in the last two terms we should factor out a negative in order to make the signs match the first two terms.

$$\begin{aligned}&4x^3-4x-x^2+1\\=\ &4x(x^2-1)-(x^2-1)\end{aligned}$$

So we see we have a common binomial factor of x^2-1. So we factor it out.

$$\begin{aligned}&4x(x^2-1)-(x^2-1)\\=\ &(x^2-1)(4x-1)\end{aligned}$$

Now we need to make sure that we have completely factored, so we need to factor the difference of squares x^2-1 as we have learned previously. So we get

$$\begin{aligned}&(x^2-1)(4x-1)\\=\ &(x-1)(x+1)(4x-1)\end{aligned}$$

c. This time we have some powers associated with the common binomial factor. What we need to recall is that whenever we factor out a GCF, we always choose to take out the lower power of all the options. So in this case, the lower power of the common factor is 2. So we will factor out a $(3x-1)^2$.

We need to also recall that when we factor out something with powers, that we subtract the powers as we go. So when we take the $(3x-1)^2$ out of the first term, we will have to subtract the powers to see what we have left. So we have

$$x(3x-1)^3-(3x-1)^2$$
$$=(3x-1)^2(x(3x-1)-1)$$

Now we need to just finish by simplifying by distributing. We get

$$(3x-1)^2(3x^2-x-1)$$

d. Lastly, we will proceed as we did with part c above and factor out the common binomial to its lowest power and simplify. We get

$$x(x+2)^4+2(x+2)^3$$
$$=(x+2)^3(x(x+2)+2)$$
$$=(x+2)^3(x^2+2x+2)$$

Now that we have a good feel for factoring binomial GCF's again, we need to continue to expand our factoring capabilities.

So, we want to take a look at factoring when we have negative exponents. To do so, we just need to keep in mind the concepts that we have always used for factoring. That is, as we mentioned in the first example, to determine a GCF we need to always choose the lowest power that is available.

<u>Example 2</u>:

Factor.

a. $m^{-7}n^{-3}-m^{-5}n^6+m^2n^{-5}$
b. $5x(x-2)^{-3}+3(x-2)^{-2}$
c. $2x^{-2}y(2x+3)^{-4}-x^{-1}y(2x+3)^{-2}$
d. $a^{-2}-4a^{-1}c^{-1}-21c^{-2}$

Solution:

a. To factor this polynomial, we need to first notice that the GCF is $m^{-7}n^{-5}$. This is the case because the lowest power of *m* is -7 and the lowest power of *n* is -5. So to factor these out, we need to keep in mind that we subtract the powers as we take them out of each term.

We need to be careful as we subtract these powers. Remember, the powers are subtracted as they are. Its always **preexisting power – the power being factored out**. In that order.

So for the first term, when we factor out the n^{-5} from the n^{-3}, the power we will end up with is $-3-(-5)=2$. With this in mind we get

$$m^{-7}n^{-3}-m^{-5}n^6+m^2n^{-5}$$
$$=m^{-7}n^{-5}(n^2-m^2n^{11}+m^9)$$

b. In this case we clearly have a common binomial. We treat this the same way we did in part a above but we also use the concepts from example 1 above. The GCF here must be $(x-2)^{-3}$. So we factor it out and subtract the powers as before.

$$5x(x-2)^{-3}+3(x-2)^{-2}$$
$$=(x-2)^{-3}\big(5x+3(x-2)\big)$$
$$=(x-2)^{-3}(5x+3x-6)$$
$$=(x-2)^{-3}(8x-6)$$

Since we notice that we clearly have another GCF of 2, we must factor that out as well. This gives us

$$\begin{aligned}&(x-2)^{-3}(8x-6)\\ &= 2(x-2)^{-3}(4x-3)\end{aligned}$$

c. In this case, the GCF contains both a monomial as well as a binomial. So, choosing the lower powers as always we have a GCF of $x^{-2}y(2x+3)^{-4}$. Factoring and simplifying as before we get

$$\begin{aligned}&2x^{-2}y(2x+3)^{-4} - x^{-1}y(2x+3)^{-2}\\ &= x^{-2}y(2x+3)^{-4}(2 - x(2x+3)^2)\\ &= x^{-2}y(2x+3)^{-4}\big(2 - x(4x^2+12x+9)\big)\\ &= x^{-2}y(2x+3)^{-4}(-4x^3-12x^2-9x+2)\end{aligned}$$

d. Lastly, for this example, we need to notice that this polynomial has no GCF. However, this is a trinomial and we can factor it the same way we learned how to factor trinomials earlier in the text. We simply need to use trial factors for the coefficients and make sure we place the variables in such a way that makes sense.

In this case, the variables should be placed so that each first term will have an a^{-1} and each second term will have a c^{-1}. This will ensure that the middle term will end up with an $a^{-1}c^{-1}$ while the leading term will have an a^{-2} and the last term will have a c^{-2}. So we get

$$\begin{aligned}&a^{-2} - 4a^{-1}c^{-1} - 21c^{-2}\\ &= (a^{-1} - 7c^{-1})(a^{-1} + 3c^{-1})\end{aligned}$$

Next we need to learn how to factor rational exponents. We just keep in mind all of the concepts that we have been working on so far. That is, always choosing the lowest power when factoring out a GCF and subtracting the powers in the order **preexisting power – the power being factored out**.

Example 3:

Factor.

a. $4x^{2/3} - 3x^{1/3} - 7x^{4/3}$

b. $2y(2y-1)^{1/2} - 7(2y-1)^{3/2}$

c. $4y(4y-1)^{3/5} - (4y-1)^{2/5}$

d. $4x^{2/3} + 15x^{1/3} + 9$

Solution:

a. In this case, using the concepts from above, we see the GCF must be $x^{1/3}$. So factoring and subtracting the powers properly we get

$$\begin{aligned}&4x^{2/3} - 3x^{1/3} - 7x^{4/3}\\ &= x^{1/3}(4x^{1/3} - 3 - 7x^{3/3})\\ &= x^{1/3}(4x^{1/3} - 3 - 7x)\end{aligned}$$

b. Here we clearly have a binomial GCF of $(2y-1)^{1/2}$. So we will factor as before.

$$\begin{aligned}&2y(2y-1)^{1/2} - 7(2y-1)^{3/2}\\ &= (2y-1)^{1/2}(2y - 7(2y-1))\\ &= (2y-1)^{1/2}(2y - 14y + 7)\\ &= (2y-1)^{1/2}(-12y + 7)\end{aligned}$$

c. Again, we need to factor out the common binomial and then simplify. The GCF here is $(4y-1)^{2/5}$. So we have

$$4y(4y-1)^{3/5}-(4y-1)^{2/5}$$
$$=(4y-1)^{2/5}(4y(4y-1)^{1/5}-1)$$

In this case, we cannot simplify any further so this is our answer.

d. Lastly, we again have a trinomial which we will need to factor by trial factors. For the variable, the only choice that makes sense is to place an $x^{1/3}$ in each first term. This will ensure we get an $x^{2/3}$ in the front and that the middle will have an $x^{1/3}$. We get

$$4x^{2/3}+15x^{1/3}+9$$
$$=(4x^{1/3}+3)(x^{1/3}+3)$$

Lastly, we will combine the rational exponents with negative exponents and factor expressions that contain negative rational exponents.

Example 4:

Factor.

a. $28x^{-7/3}-35x^{-4/3}-98x^{-1/3}$

b. $x(2x-3)^{1/2}-6(2x-3)^{-1/2}$

c. $x^{-2/3}(2x+3)^{1/2}+3x^{1/3}(2x+3)^{-1/2}$

Solution:

a. We simply need to combine all of the concepts at once. So choosing the GCF properly and carefully subtracting the powers as before will allow us to factor. So in this case the GCF must be $7x^{-7/3}$. Factoring out we get

$$28x^{-7/3}-35x^{-4/3}-98x^{-1/3}$$
$$=7x^{-7/3}(4-5x^{3/3}-14x^{6/3})$$
$$=7x^{-7/3}(4-5x-14x^2)$$

b. In this case, we have a common binomial GCF of $(2x-3)^{-1/2}$. Factoring out we get

$$x(2x-3)^{1/2}-6(2x-3)^{-1/2}$$
$$=(2x-3)^{-1/2}(x(2x-3)-6)$$
$$=(2x-3)^{-1/2}(2x^2-3x-6)$$

c. Lastly, we have a GCF of $x^{-2/3}(2x+3)^{-1/2}$. So factoring and subtracting powers we get

$$x^{-2/3}(2x+3)^{1/2}+3x^{1/3}(2x+3)^{-1/2}$$
$$=x^{-2/3}(2x+3)^{-1/2}((2x+3)+3x)$$
$$=x^{-2/3}(2x+3)^{-1/2}(5x+3)$$

R.7 Exercises

Factor.

1. $x^2(x+1)+2(x+1)$
2. $y(y+2)+3(y+2)$
3. $3y(2y+1)-(2y+1)$
4. $2x(4x-3)-3(4x-3)$
5. $a(a+1)^2+2(a+1)$
6. $2y(y+2)-5(y+2)^2$

7. $x(3x-2)^4-2(3x-2)^3$ 8. $b(b-1)^5-3(b-1)^3$ 9. $x^2y(x+y)-xy(x+y)^2$

10. $2x(2x+1)^3-4x(2x+1)^2$ 11. x^3+2x^2+3x+6 12. y^3-3y^2+2y-6

13. a^3+a^2-a-1

14. x^3-2x^2-x+2

15. $x^{-1}y^{-2}+3x^{-3}y^{-1}+2x^{-2}y^{-5}$

16. $a^{-3}b^{-2}-a^{-4}b^{-5}-2ab^{-3}$

17. $x^2(x-1)^{-2}-2(x-1)^{-1}$

18. $y^2(x+y)^{-3}-4(x+y)^{-2}$

19. $2x(3x-2)^{-4}-(3x-2)^{-2}$

20. $4x(2x-3)^{-3}-3(2x-3)^{-1}$

21. $x^{-2}y(x-1)^{-3}-x^{-1}y^{-2}(x-1)^{-4}$

22. $x^{-1}y^{-1}(x+y)^{-2}+x^{-2}y^{-3}(x+y)^{-3}$

23. $x^{-2}-2x^{-1}y^{-1}+y^{-2}$

24. $x^{-2}-x^{-1}y^{-1}-2y^{-2}$

25. $2a^{-2}+a^{-1}b^{-1}-3b^{-2}$

26. $4a^{-2}-4a^{-1}b^{-1}+b^{-2}$

27. $x^{1/3}-2x^{2/3}+3x^{5/3}$

28. $y^{1/6}-3y^{5/6}-7y^{7/6}$

29. $2x(2x-1)^{1/3}-3(2x-1)^{4/3}$

30. $3y(y+1)^{4/3}+2(y+1)^{1/3}$

31. $a(a+b)^{6/5}+b(a+b)^{1/5}$

32. $2x(2x-3)^{3/4}-(2x-3)^{1/4}$

33. $x(x-1)^{1/2}-(x-1)^{3/2}$

34. $2y(2y-1)^{5/2}-4(2y-1)^{3/2}$

35. $x^{2/3}+x^{1/3}-6$

36. $x^{1/2}-x^{1/4}-20$

37. $a^{2/3}+2a^{1/3}b^{1/3}+b^{2/3}$

38. $y^{2/3}-5x^{1/3}y^{1/3}+6x^{2/3}$

39. $5x^{-4/5}-10x^{-3/5}-15x^{-1/5}$

40. $36x^{-2/3}+12x^{-5/3}-9x^{-7/3}$

41. $x(x-1)^{-1/3}-2(x-1)^{-4/3}$

42. $a(2a+3)^{-1/2}+(2a+3)^{-3/2}$

43. $y(y+2)^{-7/4}+2(y+2)^{-3/4}$

44. $b(b+1)^{-2/3}-3(b+1)^{-1/3}$

45. $x^{-1/2}y(x+y)^{-1/3}-x^{1/2}y^2(x+y)^{-4/3}$

46. $x^{-1/3}(x-3)^{-1/3}-x^{-2/3}(x-3)^{-2/3}$

47. $2x^{-1/2}(2x-1)^{-5/3}-4x^{-3/2}(2x-1)^{-2/3}$

48. $6y^{-1/5}(3y-2)^{-1/6}-3y^{-6/5}(3y-2)^{-7/6}$

49. $x^{-2/3}y^{-2/3}-2x^{-1/3}y^{-1/3}+1$

50. $x^{-1/2}+2x^{-1/4}-3$