R.6 Radical Expressions and Equations

Our goal in this section is to merely provide a brief review of radicals and radical equations. For a more expanded discussion, refer to a full treatment of radicals in chapter 8.

We begin by simply reviewing the definition and simplifying of radicals.

Definition:
If $a > 0$ and *n* is positive, then $a^{1/n}$ is called the <u>nth root</u> of a. The value of $a^{1/n}$ is a number such that the nth power of the number gives you a.

<u>Example 1</u>:

Evaluate the following.

a. $49^{1/2}$ b. $8^{1/3}$ c. $(-4)^{1/2}$ d. $(-27)^{1/3}$

Solution:

a. We want a number that when we raise it to the second power we get 49. Since $7^2 = 49$, we have $49^{1/2} = 7$.

b. Here we want a number that when raised to the third power will give us 8. Since $2^3 = 8$, we have $8^{1/3} = 2$.

c. This time we need a number that when it is raised to the second power we get a –4. There is no such number since anything to the second power would be a positive. Therefore, we say $(-4)^{1/2}$ is not a real number.

d. Lastly, we want a number that we can raise to the third power and get –27. Since we have an odd power here it is possible to get a negative. Therefore, since $(-3)^3 = 27$, we have $(-27)^{1/3} = -3$.

As it turns out, all of the properties and rules that we had for exponents before will still work when the exponent is a rational number.

A more widely used notation for nth roots is the radical.

Definition:
If a is a real number, then $a^{1/n} = \sqrt[n]{a}$, *n* is called the <u>index</u>, $\sqrt{}$ is called the <u>radical symbol</u> and the expression underneath the radical is called the <u>radicand</u>.

This radical notation is going to be used frequently throughout the rest of this text.

Note: If there is no index on the radical it is assumed to be a two, we call it a <u>square root</u>.

Rule for Radicals: $a^{m/n} = a^{m \bullet 1/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$ and $a^{m/n} = a^{1/n \bullet m} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m$

So we can see from this that we can easily change between radical and exponent notation. We simply need to remember that the index is the denominator and vice versa.

Example 2:

Rewrite in the alternate notation.

a. $5^{1/2}$ b. $-3a^{2/5}$ c. $\sqrt[5]{4y^7}$ d. $2y\sqrt{x^3}$

Solution:

a. By the above definition, we know that the denominator becomes the index. However, when the index is 2, we need not write it. Therefore, $5^{1/2} = \sqrt{5}$.

b. On this example we need to be careful. Remember that the exponent only goes with the object right before it. In this case that means that the $2/5$ only goes with the a. So the –3 will remain out in front of the radical expression. Also, 5 will be the index, since it is the denominator. So we have $-3a^{2/5} = -3\sqrt[5]{a^2}$.

c. This time we are trying to go back to the exponent notation. Notice that the entire expression $4y^7$ is under the radical symbol. That means that the exponent will go to the entire expression $4y^7$. Since the index becomes the denominator we have $\sqrt[5]{4y^7} = \left(4y^7\right)^{1/5}$.

d. Lastly, we notice that the x^3 is the only part under the radical. Therefore, the 2y will not have the rational exponent on it. So since we see no index, we know it is a 2. Therefore we get $2y\sqrt{x^3} = 2yx^{3/2}$.

The reason this switching back and forth is important is because it is a very useful way of simplifying a radical.

We need to be able to simplify radical no matter how complicated. For this we need the following.

A radical is in simplest form when:
1. The radicand has no factors that have a power greater than the index. 2. No fractions are underneath the radical. 3. No radicals are in the denominator.

In order to deal with part one of the rule we will need the following property.

Product Property of Radicals
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers then, $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

Let’s take a look at simplifying radicals with a few examples.

Example 3:

Simplify.

a. $\sqrt{x^3y^6z^9}$ b. $\sqrt[3]{-216x^5y^{10}}$ c. $\sqrt[4]{64x^8y^{10}z^{15}}$

Solution:

a. Because of the nice relationship between radicals and fractional exponents, we should try to find a way to make the powers under the radical become multiples of the index. If we do so then they would easily simplify. Since there is no index shown, it is a two. So let's rewrite each variable as having a power that is a multiple of two, times whatever else we need. We do so as follows

$$\sqrt{x^3y^6z^9} = \sqrt{x^2xy^6z^8z}$$

Now we can group together all of the parts that have the "nice" powers in the front and all the extra stuff in the back.

$$\sqrt{x^2xy^6z^8z} = \sqrt{x^2y^6z^8(xz)}$$

Using the product property for radicals we get

$$\sqrt{x^2y^6z^8(xz)} = \sqrt{x^2y^6z^8}\cdot\sqrt{xz}$$

Notice that we can now simplify the front part as we did in example 1.

$$\begin{aligned}\sqrt{x^2y^6z^8}\cdot\sqrt{xz} &= \left(x^2y^6z^8\right)^{1/2}\cdot\sqrt{xz}\\ &= xy^3z^4\sqrt{xz}\end{aligned}$$

All the powers under the radical are smaller than the index and so the radical is simplified.

b. We will start by prime factoring the 216: $216 = 2^3\cdot 3^3$. Now we continue as before, that is, make everything in the radicand have a power that is a multiple of the index times the left over stuff. We proceed as follows

$$\begin{aligned}\sqrt[3]{-216x^5y^{10}} &= \sqrt[3]{-2^3\cdot 3^3x^3y^9\left(x^2y\right)}\\ &= \sqrt[3]{-2^3\cdot 3^3x^3y^9}\sqrt[3]{x^2y}\\ &= \left(-2^3\cdot 3^3x^3y^9\right)^{1/3}\sqrt[3]{x^2y}\\ &= -2\cdot 3xy^3\sqrt[3]{x^2y}\\ &= -6xy^3\sqrt[3]{x^2y}\end{aligned}$$

Notice, since the index is odd, the negative under the radical can just be carried out since we know that answer will be negative.

c. Lastly, we proceed as we have for all the other examples.

$$\begin{aligned}\sqrt[4]{64x^8y^{10}z^{15}} &= \sqrt[4]{2^6x^8y^{10}z^{15}}\\ &= \sqrt[4]{2^4x^8y^8z^{12}\left(2^2y^2z^3\right)}\\ &= \sqrt[4]{2^4x^8y^8z^{12}}\sqrt[4]{2^2y^2z^3}\\ &= \left(2^4x^8y^8z^{12}\right)^{1/4}\sqrt[4]{4y^2z^3}\\ &= 2x^2y^2z^3\sqrt[4]{4y^2z^3}\end{aligned}$$

So again, the key to the first part of simplifying radicals is to rewrite the powers under the radical as multiples of the index. The remaining parts we can deal with when we revisit division of radicals.

Therefore, next, we need to take a quick look at performing basic operations on radicals.

In general, doing operations on radicals is done in the same manner as polynomials are. For adding and subtracting, we simply need to "combine like radicals".

Definition: Like Radicals
Two radicals are like radicals if they have the same index and same radicand. That is, if everything after the coefficient is exactly the same, then the radicals are like.

Of course, in order to tell if two radicals are like, they first need to be in simplest form.

So to combine like radicals we just add or subtract coefficients, just as we did for polynomials.

Example 4:

Perform the indicated operations.

a. $3\sqrt{y}+12\sqrt{y}$

b. $\sqrt{18b}-\sqrt{75b}$

c. $2\sqrt{32x^2y^3}-xy\sqrt{98y}$

d. $2\sqrt[3]{24x^3y^4}+4x\sqrt[3]{81y^4}-3y\sqrt[3]{24x^3y}$

Solution:

a. Notice first that these are like radicals. So we simply need to add the coefficients to get

$$3\sqrt{y}+12\sqrt{y}=15\sqrt{y}$$

b. First we need to simplify each of the radicals as we learned in the last section.

$$\begin{aligned}\sqrt{18b}-\sqrt{75b}&=\sqrt{9\cdot 2b}-\sqrt{25\cdot 3b}\\&=3\sqrt{2b}-5\sqrt{3b}\end{aligned}$$

Since the radicand is different, the radicals are not like. Therefore we cannot combine them and we are finished with the problem.

c. Again, we will simplify the radicals and combine them if they are like.

$$\begin{aligned}2\sqrt{32x^2y^3}-xy\sqrt{98y}&=2\sqrt{16x^2y^2(2y)}-xy\sqrt{49(2y)}\\&=2\cdot 4xy\sqrt{2y}-7xy\sqrt{2y}\\&=8xy\sqrt{2y}-7xy\sqrt{2y}\\&=xy\sqrt{2y}\end{aligned}$$

d. Lastly, we proceed as we did above, simplifying and combining.

$$\begin{aligned}2\sqrt[3]{24x^3y^4}+4x\sqrt[3]{81y^4}-3y\sqrt[3]{24x^3y}&=2\sqrt[3]{8x^3y^3(3y)}+4x\sqrt[3]{27y^3(3y)}-3y\sqrt[3]{8x^3(3y)}\\&=4xy\sqrt[3]{3y}+12xy\sqrt[3]{3y}-6xy\sqrt[3]{3y}\\&=10xy\sqrt[3]{3y}\end{aligned}$$

For multiplying radicals we really want to use the product property discussed earlier, backward, that is as $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{a\cdot b}$. This means to multiply radicals, we simply need to multiply the coefficients together and multiply the radicands together. Then simplify as usual.

Example 5:

Multiply.

a. $\sqrt{5x^3y}\cdot\sqrt{10x^3y^4}$ b. $\sqrt[4]{36a^2b^4}\cdot\sqrt[4]{12a^5b^3}$

Solution:

a. We simply multiply the radicands and simplify.

$$\begin{aligned}\sqrt{5x^3y}\cdot\sqrt{10x^3y^4} &= \sqrt{50x^6y^5}\\ &= \sqrt{25x^6y^4(2y)}\\ &= 5x^3y^2\sqrt{2y}\end{aligned}$$

b. Again, we proceed as above.

$$\begin{aligned}\sqrt[4]{36a^2b^4}\cdot\sqrt[4]{12a^5b^3} &= \sqrt[4]{432a^7b^7}\\ &= \sqrt[4]{2^4a^4b^4\left(3^3a^3b^3\right)}\\ &= 2ab\sqrt[4]{27a^3b^3}\end{aligned}$$

Now, in order to multiply expressions containing more that one term, we will simply multiply as we did with polynomials in the past. That is, we multiply each term in the first expression by each term in the second expression, and simplify.

Example 6:

Multiply.

a. $\sqrt{y}\left(\sqrt{y}-\sqrt{5}\right)$ b. $\left(\sqrt{2x}+4\right)^2$

c. $\left(\sqrt{14}-3\right)\left(\sqrt{2}+4\right)$ d. $\left(\sqrt{2x}-3\sqrt{y}\right)\left(\sqrt{2x}+3\sqrt{y}\right)$

Solution:

a. Since we are to multiply as we did with polynomials, we need to use the distrubutive property here. We must always keep in mind, though, that to multiply radicals we multiply the radicands. So we get

$$\begin{aligned}\sqrt{y}\left(\sqrt{y}-\sqrt{5}\right) &= \sqrt{y}\sqrt{y}-\sqrt{y}\sqrt{5}\\ &= \sqrt{y^2}-\sqrt{5y}\\ &= y-\sqrt{5y}\end{aligned}$$

b. In this example we have to remember that we cannot pull an exponent though a set of parenthesis if the operation inside is addition or subtraction. Instead we need to write out binomial twice and then multiply out as we did before. That is with either the FOIL method or multiplying each term in the first expression by each term in the second. We get

$$\left(\sqrt{2x}+4\right)^2 = \left(\sqrt{2x}+4\right)\left(\sqrt{2x}+4\right)$$
$$= \sqrt{2x}\sqrt{2x} + 4\sqrt{2x} + 4\sqrt{2x} + 4\cdot 4$$
$$= \sqrt{4x^2} + 8\sqrt{2x} + 16$$
$$= 2x + 8\sqrt{2x} + 16$$

c. This time we simply proceed like we did in part b. Multiply each term in the first by each term in the second. This gives

$$\left(\sqrt{14}-3\right)\left(\sqrt{2}+4\right) = \sqrt{14}\sqrt{2} + 4\sqrt{14} - 3\sqrt{2} - 3\cdot 4$$
$$= \sqrt{28} + 4\sqrt{14} - 3\sqrt{2} - 12$$
$$= 2\sqrt{7} + 4\sqrt{14} - 3\sqrt{2} - 12$$

d. Again, proceed like we did above.

$$\left(\sqrt{2x}-3\sqrt{y}\right)\left(\sqrt{2x}+3\sqrt{y}\right) = \sqrt{2x}\sqrt{2x} + \sqrt{2x}\cdot 3\sqrt{y} - 3\sqrt{y}\sqrt{2x} - 3\sqrt{y}\cdot 3\sqrt{y}$$
$$= \sqrt{4x^2} + 3\sqrt{2xy} - 3\sqrt{2xy} - 9\sqrt{y^2}$$
$$= 2x - 9y$$

To divide we need to start with the following property.

Quotient Property of Radicals
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers then, $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

We will now need to deal with the second and third parts of the simplifying rule given earlier. It should be clear that to deal with the second part (No fractions are underneath the radical) we simply use the quotient property of radicals stated above. However, to deal with the last part is a little more complicated.

Most of the time, we will not just be able to use the quotient property and be done. Mostly, a radical is left “stranded” on the denominator. To deal with this we use a process called rationalizing the denominator. In general, we have the following rule.

To <u>rationalize the denominator</u> we multiply the numerator and denominator by something that will clear the radical from the denominator.

We will need to call upon all that we have learned thus far about radicals to do this rationalizing the denominator.

Example 7:

Simplify.

a. $\dfrac{1}{\sqrt{2x}}$ b. $\sqrt[3]{\dfrac{16z^3}{y^7}}$ c. $\dfrac{3u^2}{\sqrt[4]{4u}}$ d. $\dfrac{-3}{2-\sqrt{3}}$ e. $\dfrac{3\sqrt{x}-4\sqrt{y}}{3\sqrt{x}+4\sqrt{y}}$

Solution:

a. To rationalize the denominator we need to multiply the numerator and denominator by whatever it takes to get the radical to drop off. In this case we can use $\sqrt{2x}$. Since we can notice that $\sqrt{2x}\cdot\sqrt{2x}=\sqrt{4x^2}=2x$. So we proceed as follows

$$\frac{1}{\sqrt{2x}}\frac{\sqrt{2x}}{\sqrt{2x}}=\frac{\sqrt{2x}}{2x}$$

b. Again, we will split the radical to begin the problem. This time, however, we will need to simplify both numerator and denominator before we think about rationalizing the denominator. We proceed as follows.

$$\begin{aligned}\sqrt[3]{\frac{16z^3}{y^7}}&=\frac{\sqrt[3]{16z^3}}{\sqrt[3]{y^7}}\\&=\frac{\sqrt[3]{8z^3(2)}}{\sqrt[3]{y^6(y)}}\\&=\frac{2z\,\sqrt[3]{2}}{y^2\,\sqrt[3]{y}}\end{aligned}$$

Now we need to rationalize the denominator. Remember, we will need to multiply by whatever it takes to get the radicand to have a 3rd power (so that it will match the index). We see clearly we will need a $\sqrt[3]{y^2}$. So we get

$$\begin{aligned}\frac{2z\,\sqrt[3]{2}}{y^2\,\sqrt[3]{y}}\frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}}&=\frac{2z\,\sqrt[3]{2y^2}}{y^2\,\sqrt[3]{y^3}}\\&=\frac{2z\,\sqrt[3]{2y^2}}{y^3}\end{aligned}$$

We can do no more reducing. Therefore, the radical is completely simplified.

c. Lastly, we notice that the radical in this example does not simplify. Therefore we will merely have to rationalize the denominator and reduce. Since this time we have an index of 4, we will need to make each value in the radicand have a 4th power. We know that $4 = 2^2$. So we clearly use $\sqrt[4]{4u^3}$. We get

$$\frac{3u^2}{\sqrt[4]{4u}}\frac{\sqrt[4]{4u^3}}{\sqrt[4]{4u^3}} = \frac{3u^2\sqrt[4]{4u^3}}{\sqrt[4]{16u^4}}$$
$$= \frac{3u^2\sqrt[4]{4u^3}}{2u}$$
$$= \frac{3u\sqrt[4]{4u^3}}{2}$$

In the last step we canceled a *u* on top and bottom since they we both outside the radical. We must always make sure we do any canceling that is possible to ensure that we end up in simplest form.

d. In this case, we need to multiply the numerator and denominator by what is called the conjugate of the denominator.

Definition: Conjugates
The expressions $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$ are called <u>conjugates</u>. Conjugates always have a product that is "radical free".

So, the conjugate of the denominator is $2+\sqrt{3}$. So we get

$$\frac{-3}{\left(2-\sqrt{3}\right)}\cdot\frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)} = \frac{-6-3\sqrt{3}}{4+2\sqrt{3}-2\sqrt{3}-3}$$
$$= \frac{-6-3\sqrt{3}}{1}$$
$$= -6-3\sqrt{3}$$

Now that we have no more common factors and all radicals are simplified, the expression is completely simplified.

e. Lastly, we rationalize as we did above.

$$\frac{\left(3\sqrt{x}-4\sqrt{y}\right)}{\left(3\sqrt{x}+4\sqrt{y}\right)}\cdot\frac{\left(3\sqrt{x}-4\sqrt{y}\right)}{\left(3\sqrt{x}-4\sqrt{y}\right)} = \frac{9x-12\sqrt{xy}-12\sqrt{xy}+16y}{9x-12\sqrt{xy}+12\sqrt{xy}-16y}$$
$$= \frac{9x-24\sqrt{xy}+16y}{9x-16y}$$

Since the operation on the numerator and denominator is subtraction, we cannot cancel the 9x or the 16y. Remember, we can only cancel under the operation of multiplication. So we have fully simplified the expression.

Now, let's have a quick review of the complex numbers. Recall, that we cannot have a negative under a square root, since the square of any positive or negative number is always positive. In this section we want to find a way to deal with an expression that does have a negative under a square root.

We start with the following definition

Definition: The imaginary unit
The imaginary unit, i, is defined as $i = \sqrt{-1}$. Therefore, $i^2 = -1$.

Also, we can notice that $i^3 = i^2 \cdot i = -1 \cdot i$, $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$ and $i^5 = i^4 \cdot i = 1 \cdot i = i$.

So we can easily simplify any power of i since every 4th power is just 1. Also, we can see that we should always end up with an expression that has a power of i that is at most 1.

More importantly, since we are now dealing with a negative under a square root we need to know how to properly simplify radicals with this in mind. So we have the following property.

Property of negative square roots
$\sqrt{-c} = \sqrt{-1 \cdot c} = \sqrt{-1}\sqrt{c} = i\sqrt{c}$

We use this property to simplify square roots that contain negatives.

Example 8:

Simplify.

a. $\sqrt{-9}$ b. $\sqrt{-75}$ c. $\sqrt{25} - \sqrt{-147}$

Solution:

a. We can simply use the property to simplify as follows

$$\begin{aligned} \sqrt{-9} &= i\sqrt{9} \\ &= i \cdot 3 \\ &= 3i \end{aligned}$$

b. Again, we can "remove" the negative from under the radical by pulling it out as an i and then simplify the resulting expression. We get

$$\begin{aligned} \sqrt{-75} &= i\sqrt{75} \\ &= i\sqrt{25 \cdot 3} \\ &= i \cdot 5\sqrt{3} \\ &= 5i\sqrt{3} \end{aligned}$$

Note: Textbooks differ on the position of the i in a problem of this type. Most put the i in the back of the expression. However, we choose to put it between the coefficient and the radical to remove any possible confusion of whether or not the i is under the radical. If we were to write as $5\sqrt{3}i$ it can appear that the i is part of the radicand.

c. Again, we will pull or the i and then continue to simplify.

$$\begin{aligned} \sqrt{25} - \sqrt{-147} &= \sqrt{25} - i\sqrt{147} \\ &= \sqrt{25} - i\sqrt{49 \cdot 3} \\ &= 5 - 7i\sqrt{3} \end{aligned}$$

Since we do not have like radicals we cannot combine the remaining terms and thus are finished.

Now that we have a familiarity with the imaginary unit, we can introduce the number system which it generates.

Definition: Complex Numbers
A number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$ is called a complex number. a is called the real part and b is called the imaginary part. A complex number written with the real part is first and the imaginary part is last is in standard form.

We want to be able to perform basic operations on these complex numbers. Its actually very simple. We simply need to remember that i is really a radical and $i^2=-1$. With this in mind, we can simply add, subtract and multiply as we did in the earlier part of the chapter. We simply need to make sure that we simplify all of our powers of i.

Example 9:

Perform the operations. Put your answers in standard form.

a. $(-10+2i)+(4-7i)$ b. $(-21-50i)-(2+10i)$ c. $10i(8-6i)$

d. $(3+5i)(2-15i)$ e. $\dfrac{17i}{5+3i}$ f. $\dfrac{4-5i}{4+5i}$

Solution:

a. As we said, we can simply perform operations as we did earlier in this chapter. So that means we need to combine like radicals. In this case, the terms containing i would be like. So we get

$$\begin{aligned}(-10+2i)+(4-7i)&=-10+2i+4-7i\\&=-6-5i\end{aligned}$$

b. This time we need to start by distributing the negative, then combine the like radicals. This gives

$$\begin{aligned}(-21-50i)-(2+10i)&=-21-50i-2-10i\\&=-23-60i\end{aligned}$$

c. Now, to multiply complex numbers it is actually easier to just treat them as polynomials and then just simplify the powers of i by remembering that $i^2=-1$. So we get

$$\begin{aligned}10i(8-6i)&=80i-60i^2\\&=80i-60(-1)\\&=80i+60\\&=60+80i\end{aligned}$$

Since we wanted the answers in standard form, we needed to write the real part first followed by the imaginary part.

d. Again, we will multiply as if these were polynomials and simplify the result. Always remember $i^2=-1$. This gives

$$\begin{aligned}(3+5i)(2-15i)&=6-45i+10i-75i^2\\&=6-35i-75(-1)\\&=6-35i+75\\&=81-35i\end{aligned}$$

e. We can simplify this problem by multiplying numerator and denominator by the conjugate of the denominator as we did before. Then we just need to simplify and reduce. We proceed as follows

$$\begin{aligned}\frac{17i}{(5+3i)}\frac{(5-3i)}{(5-3i)} &= \frac{85i-51i^2}{5^2+3^2} \\ &= \frac{85i-51(-1)}{25+9} \\ &= \frac{51+85i}{34} \\ &= \frac{51}{34}+\frac{85}{34}i \\ &= \frac{3}{2}+\frac{5}{2}i\end{aligned}$$

f. Again, we simply multiply numerator and denominator by the complex conjugate of the denominator and then simplify and reduce. Notice that the conjugate is in fact the numerator, but we need not be concerned with that. We simply multiply it out as we learned before. We get

$$\begin{aligned}\frac{(4-5i)}{(4+5i)}\frac{(4-5i)}{(4-5i)} &= \frac{16-20i-20i+25i^2}{4^2+5^2} \\ &= \frac{16-40i+25(-1)}{16+25} \\ &= \frac{-9-40i}{41} \\ &= -\frac{9}{41}-\frac{40}{41}i\end{aligned}$$

Finally, we need to review radical equations.

To do this we need the following property.

n-th Power Property
If $a=b$, then $a^n=b^n$.

Basically, this property tells us we can raise both sides of any equation to any power we would like. However, we must be careful. Be careful to raise the entire side to the n-th power, and **we must always check our answers** to avoid extraneous solutions.

<u>Example 10</u>:

Solve.

a. $\sqrt{5x-4}=9$ b. $\sqrt[4]{3-x}=-1$ c. $\sqrt{x+1}=2-\sqrt{x}$

d. $\sqrt{2x+5}-\sqrt{3x-2}=1$

Solution:

a. We can use the n-th power property to get rid of the radical. Since we have a square root, we should use the second power, i.e. we should square both sides of the equation. Then we can solve like usual. We get

$$\sqrt{5x-4}=9$$
$$\left(\sqrt{5x-4}\right)^2=(9)^2$$
$$5x-4=81$$
$$5x=85$$
$$x=17$$

Now we must check our answer. We do this in the original equation. If the value does not check, then we eliminate it and would have no solution. We get

$$\sqrt{5(17)-4}=9$$
$$\sqrt{85-4}=9$$
$$\sqrt{81}=9$$
$$9=9$$

Since the value checks, our solution is set $\{17\}$.

b. Finally, we need to raise both sides to the 4th power since we have a 4th root. We get

$$\sqrt[4]{3-x}=-1$$
$$\left(\sqrt[4]{3-x}\right)^4=(-1)^4$$
$$3-x=1$$
$$-x=-2$$
$$x=2$$

Check:

$$\sqrt[4]{3-2}=-1$$
$$\sqrt[4]{1}=-1$$
$$1\neq -1$$

Since it does not check, 2 can not be a solution to the equation. Therefore, we have the equation has no solution.

c. We can begin by squaring both sides. Must, however be very careful with squaring the right side since it has two terms. It generally makes it easier to square properly by writing the entire side out twice and then multiplying as we learned in section 8.4. We get

$$\sqrt{x+1}=2-\sqrt{x}$$
$$\left(\sqrt{x+1}\right)^2=\left(2-\sqrt{x}\right)^2$$
$$x+1=\left(2-\sqrt{x}\right)\left(2-\sqrt{x}\right)$$
$$x+1=4-2\sqrt{x}-2\sqrt{x}+\left(\sqrt{x}\right)^2$$
$$x+1=4-4\sqrt{x}+x$$

Notice that we are left with another equation that has a radical. Therefore, we need to isolate again and square again. In this case however, it is easier to simply isolate the term containing the radical and the carefully square both sides. We can then continue solving as usual.

$$\begin{aligned} x+1 &= 4-4\sqrt{x}+x \\ -x-4 \quad & -4 \qquad\quad -x \\ -3 &= -4\sqrt{x} \\ (-3)^2 &= \left(-4\sqrt{x}\right)^2 \\ 9 &= 16x \\ \tfrac{9}{16} &= x \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{\tfrac{9}{16}+1} &= 2-\sqrt{\tfrac{9}{16}} \\ \sqrt{\tfrac{25}{16}} &= 2-\tfrac{3}{4} \\ \tfrac{5}{4} &= \tfrac{5}{4} \end{aligned}$$

Since the solution checks, the solution set is $\left\{\frac{9}{16}\right\}$.

d. Finally, we must start by isolating one of the two radicals in this equation. We will choose to isolate the positve one. Generally we isolate the more complicated radical in order to eliminate it first. However, either could be isolated and still produce the correct solutions. Once we have isolated the radical then we can square both sides very carefully and then continue as usual. We get

$$\begin{aligned} \sqrt{2x+5}-\sqrt{3x-2} &= 1 \\ \left(\sqrt{2x+5}\right)^2 &= \left(\sqrt{3x-2}+1\right)^2 \\ 2x+5 &= \left(\sqrt{3x-2}+1\right)\left(\sqrt{3x-2}+1\right) \\ 2x+5 &= \left(\sqrt{3x-2}\right)^2+\sqrt{3x-2}+\sqrt{3x-2}+1 \\ 2x+5 &= 3x-2+2\sqrt{3x-2}+1 \\ 2x+5 &= 3x-1+2\sqrt{3x-2} \\ -x+6 &= 2\sqrt{3x-2} \\ (-x+6)^2 &= \left(2\sqrt{3x-2}\right)^2 \\ (-x+6)(-x+6) &= 4(3x-2) \\ x^2-6x-6x+36 &= 12x-8 \\ x^2-12x+36 &= 12x-8 \end{aligned}$$

Recall, to solve an equation that has a squared term we must solve by factoring. That is, we get all terms on one side, factor and set each factor to zero. This gives

$$x^2-12x+36=12x-8$$
$$x^2-24x+44=0$$
$$(x-22)(x-2)=0$$
$$x-22=0 \quad and \quad x-2=0$$
$$x=22 \qquad\qquad x=2$$

Now we must check both answers and keep only those which work
Check:

$$\sqrt{2(22)+5}-\sqrt{3(22)-2}=1 \qquad \sqrt{2(2)+5}-\sqrt{3(2)-2}=1$$
$$\sqrt{44+5}-\sqrt{66-2}=1 \qquad \sqrt{4+5}-\sqrt{6-2}=1$$
$$\sqrt{49}-\sqrt{64}=1 \qquad \sqrt{9}-\sqrt{4}=1$$
$$7-8=1 \qquad 3-2=1$$
$$-1\neq 1 \qquad 1=1$$

So, 22 does not check but 2 does. Therefore, the 22 is an extraneous solution. So our solution set is $\{2\}$.

R.6 Exercises

Evaluate.

1. $4^{1/2}$
2. $16^{1/2}$
3. $(-8)^{1/3}$
4. $(-64)^{1/3}$
5. $1^{1/5}$
6. $(-1)^{1/7}$
7. $16^{3/4}$
8. $81^{3/4}$
9. $(-36)^{3/2}$
10. $(-9)^{5/2}$
11. $\left(\frac{1}{32}\right)^{2/5}$
12. $\left(\frac{25}{9}\right)^{3/2}$

Simplify. Assume all variables represent positive values.

13. $\sqrt{x^2}$
14. $\sqrt[3]{y^3}$
15. $\sqrt[3]{a^9}$
16. $\sqrt{b^4}$
17. $\sqrt{8}$
18. $\sqrt{12}$
19. $\sqrt{18}$
20. $\sqrt{158}$
21. $\sqrt{x^3y^5}$
22. $\sqrt{x^7y^4}$
23. $\sqrt[3]{a^6b^{10}}$
24. $\sqrt[3]{x^2y^{14}}$
25. $\sqrt[4]{48x^{13}y^{15}}$
26. $\sqrt[4]{144x^{10}y^7}$
27. $\sqrt[3]{-100x^{10}y^{10}z^{20}}$
28. $\sqrt[3]{-250x^{10}y^9z^8}$
29. $\sqrt{8}+\sqrt{18}$
30. $\sqrt{27}+2\sqrt{12}$
31. $\sqrt[3]{54x}-\sqrt[3]{16x}$
32. $2\sqrt[3]{6x^2y}-\sqrt[3]{48x^2y}$
33. $x\sqrt{9xy^2}+4\sqrt{x^3y^2}$
34. $5b\sqrt{18a^2b}+a\sqrt{8b^3}$
35. $\sqrt[4]{48}-\sqrt[4]{243}$
36. $\sqrt[4]{32t^5}-t\sqrt[4]{192t}$
37. $\sqrt[3]{16t^4}-t\sqrt[3]{54t}$
38. $\sqrt[3]{6x^5}-\sqrt[3]{48x^5}$
39. $\sqrt[4]{27a^2b^3}\sqrt[4]{9ab}$
40. $\sqrt[4]{8x^6y}\sqrt[4]{6x^2y^3}$
41. $\left(x\sqrt{y}+\sqrt{z}\right)^2$
42. $\left(\sqrt{2a}+\sqrt{2b}\right)^2$
43. $\left(\sqrt{a}-\sqrt{4b}\right)^2$
44. $\left(\sqrt{9x}+1\right)^2$
45. $\left(\sqrt{x}+2\right)\left(2\sqrt{x}-3\right)$
46. $\left(2-\sqrt{a}\right)\left(7+4\sqrt{a}\right)$
47. $\left(3+\sqrt{2}\right)\left(2-\sqrt{8}\right)$
48. $\left(1-\sqrt{27}\right)\left(1+\sqrt{3}\right)$

49. $\left(\sqrt{\alpha}+\beta\sqrt{\chi}\right)\left(\sqrt{\alpha}-2\beta\sqrt{\chi}\right)$

50. $\left(3\sqrt{n}+4\sqrt{m}\right)\left(\sqrt{n}-\sqrt{m}\right)$

51. $\left(2\sqrt{p}-3\sqrt{q}\right)\left(\sqrt{p}+2\sqrt{q}\right)$

52. $\left(7\sqrt{x}-3\sqrt{y}\right)\left(2\sqrt{x}+\sqrt{y}\right)$

53. $\sqrt[4]{\dfrac{y^5}{9x}}$

54. $\sqrt[4]{\dfrac{5}{4y}}$

55. $\sqrt[5]{\dfrac{2x^2}{3y^3}}$

56. $\sqrt[5]{\dfrac{15b^3}{4a^2}}$

57. $\dfrac{7}{\sqrt{2}-3}$

58. $\dfrac{5}{2+\sqrt{2}}$

59. $\dfrac{x}{\sqrt{x}+1}$

60. $\dfrac{2y}{\sqrt{2y}-1}$

61. $\dfrac{1}{\sqrt{2}-\sqrt{3}}$

62. $\dfrac{5}{\sqrt{5}-\sqrt{2}}$

63. $\dfrac{5}{\sqrt{x}-2\sqrt{y}}$

64. $\dfrac{6}{\sqrt{t}+2\sqrt{u}}$

65. $(17+5i)+(18-5i)$

66. $(20-4i)+(30+4i)$

67. $(15-3i)-(14-3i)$

68. $(12-5i)-(13-5i)$

69. $(-7+4i)(4-i)$

70. $(-5+2i)(6-5i)$

71. $(5-2i)(4+3i)$

72. $(6-3i)(4+5i)$

73. $\dfrac{i}{1+i}$

74. $\dfrac{3i}{3-i}$

75. $\dfrac{2+3i}{1+2i}$

76. $\dfrac{4-3i}{1+i}$

Solve.

77. $\sqrt{3x-5}-2=3$

78. $\sqrt{2x-1}-8=-5$

79. $\sqrt[3]{4x-3}-2=3$

80. $\sqrt[3]{1-3x}+5=3$

81. $\sqrt[3]{x^2+4}-2=0$

82. $\sqrt[3]{x^2+2}-3=0$

83. $4\sqrt{x+1}-x=1$

84. $3\sqrt{x-2}+2=x$

85. $\sqrt{x}+2=x$

86. $x+3\sqrt{x-2}=12$

87. $\sqrt{2+9b}-1=3\sqrt{b}$

88. $\sqrt{x+1}=2-\sqrt{x}$

89. $3+\sqrt{z-6}=\sqrt{z+9}$

90. $\sqrt{4x-3}=2+\sqrt{2x-5}$

91. $\sqrt{5t}=1+\sqrt{5(t-1)}$

92. $\sqrt{2x+4}=3-\sqrt{2x}$