R.5 Rational Expressions and Equations

In this section we want to conclude our review of the ideas we need from elementary algebra. Other important topics to review will be included later.

Here we want to review rational expressions. Recall a rational expression is just any fraction that contains a polynomial. This includes all fractions with or without variables.

We want to start by simply reducing rational expressions. This merely requires factoring and canceling any common factors.

Example 1:

Simplify.

a. $\frac{15a^2x^3}{35ax^4}$

b. $\frac{3x^3-12x}{6x^2-24x+24}$

Solution:

a. To simplify this rational expression we simply need to reduce the number part and cancel any variables that we can. We get

$$\frac{15a^2x^3}{35ax^4}=\frac{3a}{7x}$$

b. This time we need to factor the numerator and denominator completely first. Then we cancel any common factors. This gives us

$$\frac{3x^2-12x}{6x^2-24x+24}=\frac{3x(x^2-4)}{6(x^2-4x+4)}$$

$$=\frac{3x(x-2)(x+2)}{6(x-2)(x-2)}$$

$$=\frac{x(x+2)}{2(x-2)}$$

Next we want to perform all the basic operations on rational expressions, that is, adding, subtraction, multiplying and dividing rational expressions.

Recall that the techniques that we use in order to do this are the same techniques we used for regular fractions. That is, to add and subtract, we need to have the LCD. To multiply we simply multiply straight across (we actually factor and cancel first) and for dividing we invert the second fraction and then multiply.

Example 2:

Perform the operations.

a. $\frac{x^2-36}{x^2+12x+36}\cdot\frac{2x+12}{x-6}$

b. $\frac{x^2+16x+64}{x^2+9x+8}\div\frac{x^2-64}{x^2-1}$

c. $\frac{4x-1}{x^2+x-20}+\frac{3x+1}{x+5}$

d. $\frac{x}{x+2}-\frac{3x}{4x-1}$

Solution:

a. To multiply we need only factor all numerators and denominator and cancel any common factors. We get

$$\frac{x^2-36}{x^2+12x+36}\cdot\frac{2x+12}{x-6}=\frac{(x-6)(x+6)}{(x+6)(x+6)}\cdot\frac{2(x+6)}{x-6}$$
$$=\frac{2}{1}$$
$$=2$$

b. For division, we need to invert the second fraction (i.e. flip it upside down), then multiply as usual by factoring and canceling. We get

$$\frac{x^2+16x+64}{x^2+9x+8}\div\frac{x^2-64}{x^2-1}=\frac{x^2+16x+64}{x^2+9x+8}\cdot\frac{x^2-1}{x^2-64}$$

$$=\frac{(x+8)(x+8)}{(x+8)(x+1)}\cdot\frac{(x-1)(x+1)}{(x-8)(x+8)}$$

$$=\frac{x-1}{x-8}$$

c. To add rational expression we will need to find the LCD. So first we should factor all denominators. This gives

$$\frac{4x-1}{x^2+x-20}+\frac{3x+1}{x+5}=\frac{4x-1}{(x+5)(x-4)}+\frac{3x+1}{x+5}$$

Clearly the LCD is $(x+5)(x-4)$. We need each denominator to have the LCD so we can add the numerators. To accomplish this, we need to multiply the numerator and denominator by whatever it takes to get each fraction to have the LCD. Clearly the first fraction needs nothing. However, the second fraction needs an x – 4. So we once we do this we can just add the numerators and reduce. We proceed as follows

$$\frac{4x-1}{(x+5)(x-4)}+\frac{3x+1}{x+5}\cdot\frac{x-4}{x-4}=\frac{4x-1}{(x+5)(x-4)}+\frac{(3x+1)(x-4)}{(x+5)(x-4)}$$

$$=\frac{4x-1+(3x+1)(x-4)}{(x+5)(x-4)}$$

$$=\frac{4x-1+3x^2-11x-4}{(x+5)(x-4)}$$

$$=\frac{3x^2-7x-5}{(x+5)(x-4)}$$

d. Similarly, we can see the LCD here is $(x+2)(4x-1)$. So we multiply each fraction on numerator and denominator by what it is missing from the LCD. We then subtract numerators and simplify. We get

$$\frac{x}{x+2}-\frac{3x}{4x-1}=\frac{x}{x+2}\cdot\frac{4x-1}{4x-1}-\frac{3x}{4x-1}\cdot\frac{x+2}{x+2}$$
$$=\frac{x(4x-1)}{(x+2)(4x-1)}-\frac{3x(x+2)}{(4x-1)(x+2)}$$

$$= \frac{x(4x-1)-3x(x+2)}{(x+2)(4x-1)}$$

$$= \frac{4x^2 - x - 3x^2 - 6x}{(x+2)(4x-1)}$$

$$= \frac{x^2 - 7x}{(x+2)(4x-1)}$$

$$= \frac{x(x-7)}{(x+2)(4x-1)}$$

Also, when dealing with rational expressions, sometimes we have to deal with fractions inside fractions. We call these <u>complex</u> (or compound) <u>fractions</u>. We simplify them by multiplying numerator and denominator by the LCD of all the fractions in the expression.

<u>Example 3</u>:

Simplify.

a. $\dfrac{\frac{4}{x^2}+\frac{3}{x}}{\frac{2}{x^2}-\frac{5}{x}}$ b. $\dfrac{\frac{6}{x-5}+7}{\frac{8}{x-5}-\frac{9}{x+3}}$

Solution:

a. To simplify this expression, we will multiply numerator and denominator by the LCD of all the fraction contained in the expression. That is, x^2 here. Then we reduce and simplify. We get

$$\frac{\left(\frac{4}{x^2}+\frac{3}{x}\right)x^2}{\left(\frac{2}{x^2}-\frac{5}{x}\right)x^2} = \frac{\frac{4x^2}{x^2}+\frac{3x^2}{x}}{\frac{2x^2}{x^2}-\frac{5x^2}{x}}$$

$$= \frac{4+3x}{2-5x}$$

So since the rational expression is fully reduced, we are finished.

b. This time the LCD is $(x-5)(x+3)$. So multiplying on numerator and denominator gives us

$$\frac{\left(\frac{6}{x-5}+7\right)(x-5)(x+3)}{\left(\frac{8}{x-5}-\frac{9}{x+3}\right)(x-5)(x+3)} = \frac{\frac{6(x-5)(x+3)}{x-5}+7(x-5)(x+3)}{\frac{8(x-5)(x+3)}{x-5}-\frac{9(x-5)(x+3)}{x+3}}$$

$$= \frac{6(x+3)+7(x-5)(x+3)}{8(x+3)-9(x-5)}$$

$$= \frac{6x + 18 + 7x^2 - 14x - 105}{8x + 24 - 9x + 45}$$
$$= \frac{7x^2 - 8x - 87}{-x + 69}$$

Next we want to remember how to solve an equation that contains a rational expression and also how to solve equations that contain more than one variable, called a literal equation.

To solve rational equations we simply multiply both sides of the equation by the LCD. This will "clear" the fractions. We must, though, remember to check our answers, since we cannot ever have a denominator equal to zero. If one of our values makes the original equation have a denominator of zero, we must omit it and only keep those solutions that check.

To solve literal equations, we just use the same techniques we did before for solving equations. Its just that now we end up with an expression for a solution, not just a number.

Example 4:

Solve.

a. $\frac{x}{x+4} + 4 = \frac{-4}{x+4}$

b. $\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2 - 6x + 8}$

c. $P = n(P_2 - P_1) - c$ solve for P_2

d. $a = \frac{1}{b} - \frac{1}{c}$ solve for b

Solution:

a. So to solve the equation we will start by clearing the fractions. So we multiply both sides by the LCD, which is x + 4 here. Then we solve the equation that is left. We get

$$(x+4)\left(\frac{x}{x+4} + 4\right) = \left(\frac{-4}{x+4}\right)(x+4)$$
$$\frac{x(x+4)}{x+4} + 4(x+4) = \frac{-4(x+4)}{x+4}$$
$$x + 4(x+4) = -4$$
$$x + 4x + 16 = -4$$
$$5x + 16 = -4$$
$$5x = -20$$
$$x = -4$$

Now we must check to see if this value gives us a zero in any denominator of the original equation. Clearly, we get a zero for every denominator. Therefore the solution must be thrown out. So the equation has no solution.

b. Again we will multiply both sides by the LCD. So we must start by factoring every denominator to determine the LCD. Once we have done that we will continue as we did above. We get

$$\frac{x}{x-2}+\frac{1}{x-4}=\frac{2}{x^2-6x+8}$$

$$(x-2)(x-4)\left(\frac{x}{x-2}+\frac{1}{x-4}\right)=\left(\frac{2}{(x-2)(x-4)}\right)(x-2)(x-4)$$ LCD is (x-2)(x-4)

$$\frac{x(x-2)(x-4)}{x-2}+\frac{(x-2)(x-4)}{x-4}=\frac{2(x-2)(x-4)}{(x-2)(x-4)}$$

$$x(x-4)+(x-2)=2$$

$$x^2-4x+x-2=2$$ Solve by factoring

$$x^2-3x-4=0$$

$$(x-4)(x+1)=0$$

$$x=-1,\ 4$$

Again we need to check these answers. We see that the –1 checks ok, however, the 4 does not check. Therefore the solution set is simply $\{-1\}$.

c. So this time we have a literal equation. To solve this, we use the same techniques that we always did, it's just that we will have an answer that contains several variables. The object is to solve so that we get the variable that we are solving for alone on one side, and everything else on the other side. Since we are solving for P_2, we proceed as follows

$$P=n(P_2-P_1)-c$$ Clear the parenthesis

$$P=nP_2-nP_1-c$$

$$P+nP_1+c=nP_2$$ Get the term containing P_2 alone

Divide by n

$$P_2=\frac{P+nP_1+c}{n}$$

d. Again we proceed as we did in the last problem. However, this time we need to start by clearing the fractions. Since the LCD is bc, we get

$$bc(a)=\left(\frac{1}{b}-\frac{1}{c}\right)bc$$ Clear the fractions

$$abc=\frac{bc}{b}-\frac{bc}{c}$$

$$abc=c-b$$

Get all the terms with b to the same side

$$abc+b=c$$

Factor out the b

$$b(ac+1)=c$$

Divide by ac+1

$$b=\frac{c}{ac+1}$$

Lastly we have some applications of rational expressions. Some of the more common types are work problems and moving object problems. We use a table to collect the data and make an equation to solve. We will need the following formulas: distance = rate x time ($d = r \cdot t$) and rate of work x time worked = part of task completed ($r \cdot t = pc$)

Example 5:

Two painters, working together, can paint a house in 10 hours. Working alone, the first painter can paint the house in 15 hours. How long would it take for the second painter to paint the house working alone?

Solution:
Since this is a problem involving work, we use the formula $r \cdot t = pc$. First notice that if the first painter alone takes 15 hours, then his rate of work is $\frac{1}{15}$ of the house painted per hour. In a similar fashion, if the second painter alone takes x hours, then his rate of work is $\frac{1}{x}$ of the house painted per hour. So the since they work together for 10 we can construct the following table to summarize our results.

	r $\cdot$	t $=$	pc
Painter 1	$\frac{1}{15}$	10	$\frac{10}{15}$
Painter 2	$\frac{1}{x}$	10	$\frac{10}{x}$

So, since they get the entire house painted in 10 hours, we know that the total part that was completed would be 1 (if they only painted half of the house then it would be ½ and if they painted 2 houses it would be 2 etc.). So therefore since the last column of the table is the part completed by each painter, we can get the equation

$$\frac{10}{15} + \frac{10}{x} = 1$$

Now we simply solve for x to find the time required for the second painter alone. We get

$$\frac{10}{15} + \frac{10}{x} = 1 \qquad \text{Reduce the fraction}$$

$$3x\left(\frac{2}{3} + \frac{10}{x}\right) = (1)3x \qquad \text{Multiply by the LCD 3x}$$

$$\frac{6x}{3} + \frac{30x}{x} = 3x \qquad \text{Reduce and solve}$$

$$2x + 30 = 3x$$

$$x = 30$$

So, working alone, the second painter would take 30 hours to paint the house.

Example 6:

A canoeist can paddle 8 mph in still water. Traveling with the current, the canoe traveled 30 mi in the same amount of time in which it traveled 18 mi against the current. Find the rate of the current.

Solution:
This problem deals with a moving object. So we will need to use the formula $d = r \cdot t$. Since we are looking for the rate of the current, lets call it *c*. Therefore, the rate that the canoeist would travel with the current would be 8 + *c* (since he goes 8 mph in still water, his speed is increased

by exactly the speed of the current). Likewise the rate against the current would be 8 – *c*. So we can use this information to start filling in the table as follows.

	r ·	t =	d
With the current	$8+c$		30
Against the current	$8-c$		18

So we still need to fill in the time column. However, we know that if $d = r \cdot t$, then $t = \frac{d}{r}$. So using this fact we can get the time column by taking what's in the distance column and put it over what's in the rate column. We get

	r ·	t =	d
With the current	$8+c$	$\frac{30}{8+c}$	30
Against the current	$8-c$	$\frac{18}{8-c}$	18

Now we are ready to make an equation. Since it the problem says the canoe traveled 30 mi with the current **in the same amount of time** in which it traveled 18 mi against the current we can see that the time for each of these must be equal. So we get the equation $\frac{30}{8+c} = \frac{18}{8-c}$. Now we solve.

$$(8-c)(8+c)\left(\frac{30}{8+c}\right) = \left(\frac{18}{8-c}\right)(8-c)(8+c) \quad \text{Multiply by the LCD (8+c)(8-c)}$$

$$30(8-c) = 18(8+c) \quad \text{Reduce and solve}$$

$$240 - 30c = 144 + 18c$$

$$48c = 96$$

$$c = 2$$

So the current was 2 mph.

R.5 Exercises

Simplify.

1. $\frac{16x^2y}{24xy^3}$
2. $\frac{x^2-3x}{2x-6}$
3. $\frac{x^2+7x-8}{x^2+6x-7}$
4. $\frac{x^2-1}{2x^2+x-3}$
5. $\frac{6-x}{x^2+3x-54}$
6. $\frac{x-4}{16-x^2}$
7. $\frac{20-30x}{3x^2-2x}$

Perform the indicated operations.

8. $\frac{4x^2}{9y^3} \cdot \frac{3y^5}{4x}$
9. $\frac{3x-6}{5x-20} \cdot \frac{10x-40}{27x-54}$
10. $\frac{x^2-5x-14}{x+8} \cdot \frac{x^2-64}{x^2-6x-16}$
11. $\frac{x-7}{5x^2+25x} \cdot \frac{x^2+12x+35}{x^2-49}$

12. $\dfrac{x^4y}{x^2-x-6} \div \dfrac{x^3y}{x^2-4}$

13. $\dfrac{25x^2-4}{x+3} \div \dfrac{5x^2-13x-6}{x^2-9}$

14. $\dfrac{x^2+8x+16}{x^2-16} \div \dfrac{x^2-3x-28}{x-7}$

15. $\dfrac{2x}{x+7} - \dfrac{5x-8}{x+7}$

16. $\dfrac{3x^2}{x^2-1} - \dfrac{x+4}{x^2-1}$

17. $\dfrac{2x+3}{x^2-x-30} - \dfrac{x-2}{x^2-x-30}$

18. $\dfrac{1}{2x} - \dfrac{5}{4x} + \dfrac{7}{6x}$

19. $\dfrac{3x-2}{12x} - \dfrac{x-3}{18x}$

20. $\dfrac{3}{x+6} - \dfrac{4}{x-3}$

21. $\dfrac{7}{x^2-3x-40} + \dfrac{3}{x+5}$

22. $\dfrac{9}{x} + \dfrac{2}{x-6}$

23. $\dfrac{4}{x^2-7x-18} - \dfrac{1}{x^2-4}$

24. $\dfrac{7}{x^2+5x} + \dfrac{2x}{x+5}$

25. $\dfrac{1}{x+1} + \dfrac{x}{x-6} - \dfrac{5x-2}{x^2-5x-6}$

Simplify.

26. $\dfrac{1-\dfrac{4}{x^2}}{1+\dfrac{2}{x}}$

27. $\dfrac{\dfrac{1}{3}-\dfrac{1}{x}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

28. $\dfrac{\dfrac{8}{x}+\dfrac{8}{x^2}}{\dfrac{1}{x^2}-1}$

29. $\dfrac{1-\dfrac{7}{x}+\dfrac{12}{x^2}}{1+\dfrac{1}{x}-\dfrac{20}{x^2}}$

30. $\dfrac{\dfrac{3}{x+1}+\dfrac{1}{x}}{\dfrac{2}{x+1}+\dfrac{3}{x}}$

31. $\dfrac{1+\dfrac{4}{x}+\dfrac{4}{x^2}}{1-\dfrac{2}{x}-\dfrac{8}{x^2}}$

32. $\dfrac{x+\dfrac{1}{x-2}}{1+\dfrac{1}{x-2}}$

33. $\dfrac{x+3-\dfrac{20}{x-5}}{x+8+\dfrac{30}{x-5}}$

34. $\dfrac{x-8+\dfrac{20}{x+4}}{x-10+\dfrac{24}{x+4}}$

Solve.

35. $\dfrac{3}{8}x+\dfrac{1}{2}=\dfrac{1}{4}x+2$

36. $\dfrac{1}{3}x+2=\dfrac{1}{2}x-1$

37. $\dfrac{5}{6}x-\dfrac{2}{3}=\dfrac{3}{4}x+2$

38. $\dfrac{5}{8x}-\dfrac{1}{2}=\dfrac{7}{6x}$

39. $\dfrac{4}{x^2-5x-36}=\dfrac{-1}{x+4}$

40. $\dfrac{3}{x-7}=\dfrac{2}{4x+1}$

41. $\dfrac{x}{x+4}=\dfrac{2}{x}$

42. $\dfrac{8}{x^2-6x+8}=\dfrac{1}{x^2-16}$

43. $x-\dfrac{6}{x-3}=\dfrac{2x}{x-3}$

44. $\dfrac{1}{x-3}+\dfrac{2}{x+4}=\dfrac{6}{x^2+x-12}$

45. $\dfrac{9}{6x^2+x-2}=\dfrac{5}{2x-1}-\dfrac{7}{3x+2}$

46. $I=prt$ solve for r

47. $p=2L+2W$ solve for W

48. $V=gt+k$ solve for t

49. $A=\dfrac{1}{2}h(b+c)$ solve for c

50. $A=\frac{1}{2}h(b+c)$ solve for h

51. $L=\frac{E}{R+K}$ solve for R

52. $s=a+(n-1)d$ solve for n

53. $s=a+(n-1)d$ solve for d

54. $N=C-rC$ solve for C

55. $L=\frac{1}{3}(act+a)$ solve for a

56. Machine #1 can do a job in 8 hrs. Machine #2 can do the same job in 6 hrs. How long will it take both machines to do the job working together?

57. Machine #1 can do a job in 5hrs. Operating with machine #2 they can do the job in 3 hrs. How long will it take machine #2 to do the job alone?

58. A small water pipe takes three times longer to fill a tank than does a large water pipe. With both pipes open, it takes 4 hrs to fill the tank. Find the time it would take the small pipe, working alone, to fill the tank.

59. A small copier takes two times longer than a large copier to complete a job. With both copiers, operating together, it takes 18 minutes to complete the same job. How long will it take the large copier to complete the job operating alone?

60. Two computer printers that work at the same rate are working together to print the payroll checks for a large corporation. After they work together for 4 hrs, one of the printers breaks down. The second printer requires 3 hrs more to complete the payroll checks. Find the time it would take one printer, working alone, to print the payroll.

61. An account executive traveled 900 mi on a corporate jet and then an additional 90 mi by helicopter. The rate of the jet was five times the rate of the helicopter. The entire trip took 3 hours. Find the rate of the jet.

62. The rate of a motorcycle is 40 mph greater than the rate of a bicycle. The motorcycle travels 150 mi in the same amount of time it takes the bicycle to travel 30 mi. Find the rate of the motorcycle.

63. A cyclist rode the first 11 mi of a trip at a constant rate. For the next 16 mi, the cyclist reduced the speed by 3 mph. The total time for the 27-mile trip was 3 h. Find the rate of the cyclist on the 16 mi leg of the trip.

64. A small plane can fly 140 mph in calm air. Traveling with the wind, the plane can fly 425 mi in the same amount of time in which it can fly 275 mi against the wind. Find the rate of the wind.

65. The speed of a boat in still water is 28 mph. The boat traveled 70 mi down a river in the same amount of time in which it traveled 42 mi up the river. Find the rate of the rivers current.