R.3 Absolute Values

We begin this section by recalling the following definition.

Definition: Absolute Value

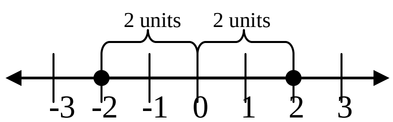
The absolute value of a number is the distance that the number is from zero. The absolute value of *x* is written $|x|$.

We want to do two major things in this section: solve absolute value equations and solve absolute value inequalities.

Lets deal with absolute value equations first.

Consider the equation $|x| = 2$.

By definition of absolute value, we want all the *x* values that have a distance from zero that equals 2 units. Graphically we can see



So clearly the solutions to our equation are $x = 2$ and $x = -2$.

Therefore we make the following generalization.

Solving an absolute value equation

If $|x| = a$, then $x = a$ and $x = -a$.

We use this to solve. Note, however, the *x* in the above property usually represents an entire expression. Also, **we need to always start by isolating the absolute value expression before using this property**.

<u>Example 1</u>:

Solve the following.

a. $|x+5| = 2$ b. $|5x-5|+1=1$ c. $8-|3x-2|=5$

Solution:

a. Since the absolute value is already isolated we can simply use the above property to eliminate the absolute value from the expression. This gives us

$$|x+5| = 2$$

$$\begin{array}{ccc} x+5=2 & & x+5=-2 \\ & and & \\ x=-3 & & x=-7 \end{array}$$

So the solution set is $\{-7, -3\}$.

b. We will start by isolating the absolute value expression and then we will continue by using the property.

$$|5x-5|+1=1$$
$$|5x-5|=0$$
$$5x-5=0 \quad and \quad 5x-5=-0$$

However, since 0 = -0, we only end up with one case, $5x-5=0$. So we solve accordingly.

$$5x-5=0$$
$$5x=5$$
$$x=1$$

So the solution set is $\{1\}$.

c. Lastly, we again isolate the absolute value and then use the property to break it apart. We get

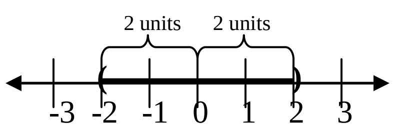
$$8-|3x-2|=5$$
$$-|3x-2|=-3$$
$$|3x-2|=3$$
$$3x-2=3 \quad and \quad 3x-2=-3$$
$$3x=5 \quad and \quad 3x=-1$$
$$x=\tfrac{5}{3} \quad and \quad x=-\tfrac{1}{3}$$

So the solution set is $\left\{\tfrac{5}{3}, -\tfrac{1}{3}\right\}$.

Next we turn our attention to absolute value inequalities.

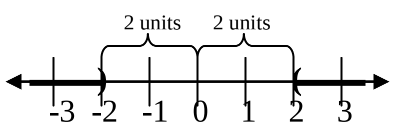
Consider the inequalities $|x|<2$ and $|x|>2$.

For $|x|<2$, by definition of absolute value, we want all the *x* values that have a distance from zero that is less than 2 units. Graphically we can see



So clearly the solution to this inequality is $(-2,\ 2)$ or in inequality notation $-2<x<2$.

Similarly, for $|x|>2$, by definition of absolute value, we want all the *x* values that have a distance from zero that is more than 2 units. Graphically we can see



So clearly the solution to this inequality is $(-\infty,\ -2)\cup(2,\ \infty)$ or in inequality notation $x<-2 \text{ or } x>2$.

We use this example to generate the following

Solving an absolute value inequality

1. $|x| < a$ if and only if $-a < x < a$, similarly for $\leq$.
2. $|x| > a$ if and only if $x < -a \text{ or } x > a$, similarly for $\geq$.

Again, the *x* in the above property usually represents an entire expression and **we need to always start by isolating the absolute value expression**.

Also, we must always be careful to remember, that anytime we multiply or divide an inequality by a negative, we must "flip" the inequality symbol.

Example 2:

Solve and graph the solution. Put your answer in interval notation.

a. $|x-3| > 5$ b. $5 \cdot |8-x| \leq 25$ c. $|7x-3|+1<12$ d. $12-|3x+4| \leq 7$

Solution:

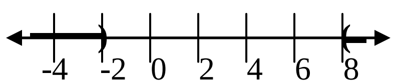
a. Since the absolute value is already isolated we can use the above property to solve. We use the second part (i.e. an "or" compound inequality) as follows

$$|x-3| > 5$$

$$x-3 < -5 \quad or \quad x-3 > 5$$

$$x < -2 \quad or \quad x > 8$$

In the last section we learned how to properly graph this kind of inequality. We simply graph both pieces and we get to keep all of it since we have the word "or". We get



So our solution set is $(-\infty, -2) \cup (8, \infty)$.

b. We start by dividing both sides by 5 to isolate the absolute value. Then we proceed by using the first part of the property above (i.e. a double inequality). We get

$$5 \cdot |8-x| \leq 25$$

$$|8-x| \leq 5$$

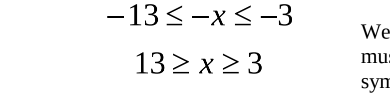
$$-5 \leq 8-x \leq 5$$

$$-13 \leq -x \leq -3$$

$$13 \geq x \geq 3$$

We divide by –1 here, so we must flip the inequality symbols

So graphing we get

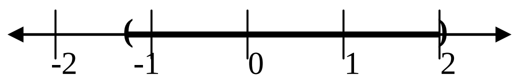


So our solution is $[3, 13]$.

c. Again, we start by isolating the absolute value and then proceed as we did above. We get

$$|7x-3|+1<12$$
$$|7x-3|<11$$
$$-11<7x-3<11$$
$$-8<7x<14$$
$$-\tfrac{8}{7}<x<2$$

Graphing we have



So the solution is $\left(-\tfrac{9}{7},\, 2\right)$.

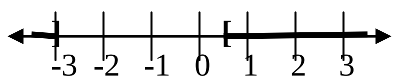
d. Finally, in this example, it is very important that we begin by isolating the absolute value. Then we can solve as we did above.

$$12-|3x+4|\le 7$$
$$-|3x+4|\le -5$$
$$|3x+4|\ge 5$$

Notice that we had to switch the inequality symbol because we divided both sides by –1. So now we have to solve as a > problem. Whereas the problem started as a < type. So we solve

$$|3x+4|\ge 5$$
$$3x+4\le -5 \quad or \quad 3x+4\ge 5$$
$$3x\le -9 \quad or \quad 3x\ge 1$$
$$x\le -3 \quad or \quad x\ge \tfrac{1}{3}$$

So we graph and write the solution in interval notation.



$(-\infty,\,-3]\cup\left[\tfrac{1}{3},\,\infty\right)$.

We must be very careful when doing absolute value inequalities. The common mistake is to try to treat them all the same. However, we can clearly see from the last example that whether the inequality symbol is a < or a > makes a big difference in the way that the problem is solved.

A < symbol always requires a double inequality and a > symbol always requires an "or" compound inequality.

Lastly, sometimes simply knowing what the absolute value really is, the distance from zero, can solve an absolute value equation or inequality. This final example will illustrate this.

<u>Example 3</u>:

Solve the following.

a. $|y|=-3$ b. $\left|\tfrac{x}{8}+1\right|<0$ c. $|x+20|>-1$

Solution:

a. Recall that the absolute value is the distance from zero. So since distance is always a positive value, we know that the absolute value can never be negative. Therefore, $|y| = -3$ must have no solution.

b. This time, we again remember that absolute value can never be negative. Since the only numbers that are less that zero are negatives we again can say that $\left|\frac{x}{8}+1\right| < 0$ has no solution.

c. Finally, we have to be careful here since we have a negative on the right side of the inequality. The temptation is to say that the inequality has no solution as above. However, since this says $|\quad| > -1$ and the absolute value is always positive (which is always greater that –1) this inequality must have all values as a solution because $|\quad| > -1$ is true regardless of what is inside the absolute value symbols. Therefore we say that the solution is all real numbers, that is $(-\infty, \infty)$.

R.3 Exercises

Solve the following.

1. $|x| = 2$
2. $|x| = 12$
3. $|x| = 7$
4. $|y| = 15$
5. $|x+2| = 3$
6. $|x+1| = 7$
7. $|x-3| = 5$
8. $|x-1| = 1$
9. $|x+1| - 1 = 0$
10. $|x-2| - 2 = 0$
11. $|2x-1| = 0$
12. $|3x-2| = 0$
13. $|x+4| + 1 = 0$
14. $|x+3| + 2 = 0$
15. $|3t+2| + 3 = 4$
16. $|5x-2| + 5 = 7$
17. $3 - |3-5x| = -2$
18. $8 - |1-3x| = -1$
19. $|1-5x| + 2 = 3$
20. $|6-5x| - 4 = 3$
21. $|2x-8| - 12 = 2$
22. $|3x-4| + 8 = 10$
23. $2 - |2x-3| = 0$
24. $7 - |3x+1| = 0$
25. $5 - |x+1| = 5$
26. $3 - |2x| = 3$
27. $8 - |3x+2| = 9$
28. $8 + |2x-5| = 5$
29. $8 - |2-3x| = 5$
30. $3 - |3+4x| = 2$

Solve and graph the solution. Put your answer in interval notation.

31. $|x| > 3$
32. $|x| < 5$
33. $|x| \le 4$
34. $|x| \ge 7$
35. $|x+1| \ge 2$
36. $|x-2| \le 1$
37. $|x-5| < 1$
38. $|x+4| > 3$
39. $|2-x| - 3 \ge 3$
40. $|3-x| - 2 \ge 2$
41. $|2x+1| - 5 < 2$
42. $|3x-2| - 4 < 0$
43. $|5x-2| - 12 \ge 0$
44. $|4x+3| + 2 \le 0$
45. $|5x+1| + 4 \le 0$
46. $2 + |3x-1| \ge 0$
47. $3|4x-2| > 6$
48. $2|7x-1| \le 4$

49. $-4|2x-1| > -8$

50. $-|7x+2| < -2$

51. $\dfrac{|x-3|}{2} \geq 3$

52. $\dfrac{|1-x|}{3} < 2$

53. $|2x+7|+5 > 1$

54. $|3x-1|+4 > 2$

55. $13+|7x-1| \leq 0$

56. $12+|5x+2| \leq 0$

57. $3+|2x-1| > 3$

58. $1+\left|x+\frac{1}{2}\right| > 1$

59. $5|3x+1|+4 \leq 4$

60. $6-2|2x-1| \geq 6$

61. $3-|2x-4| > 2$

62. $5-|5x+4| > 3$

63. $|7-2x|-20 > -3$

64. $|5-4x|-13 \geq 5$

65. $|3-7x|-20 < -3$

66. $|2-9x|-10 > 10$

67. $6-|4x+2| \leq 3$

68. $8-|3x-1| \leq 5$

69. $3-|8-2x| \geq -1$

70. $3-|3-3x| > -3$

71. $2-2|2-2x| > -2$

72. $6-3|3-2x| < 4$

73. $4-2|2-5x| \leq 2$

74. $9-3|4-2x| \leq 3$

75. $8-5|2x-5| < 3$

76. $12-2|3x-4| > 8$

77. $\frac{3}{4}-\frac{1}{2}\left|1-\frac{x}{2}\right| \geq \frac{1}{2}$

78. $\frac{1}{3}-\frac{2}{5}\left|2-\frac{x}{3}\right| \leq \frac{1}{6}$