9.5 Piecewise Defined Functions

In this section we want to take a look at a very specific type of function called a Piecewise Defined function. As it turns out, they are very useful in a variety of higher level math classes.

Let's start with the definition.

> **Definition**: **Piecewise Defined Function**- a function whose domain is broken into different sections upon which the function is different over each section.

The first thing we need to do is get a handle on evaluating a piecewise defined function. The process is quite straightforward. We simply need to decide what part of the function a value belongs in, and then just plug it into that particular portion and simplify.

Let's see this in an example.

Example 1:

Evaluate the function $f(x) = \begin{cases} x + 1 & if \ x > 0 \\ x^2 & if \ x < 0 \end{cases}$ at the given values

  a. $f(-2)$ b. $f(2)$ c. $f(3)$ d. $f(0)$

Solution:

  a. To evaluate a piecewise defined function, we must simply figure out which piece a domain value belongs to, then evaluate it in that piece.

  For this part, clearly the value of -2 belongs in the bottom part of the function. So, we just plug it into the bottom part.
  $$f(-2) = (-2)^2 = 4$$

  b. This time the value goes into the top part because 2 is in the top part of the domain.

  $$f(2) = 2 + 1 = 3$$

  c. Here, we again evaluate in the top part
  $$f(3) = 3 + 1 = 4$$

  d. Lastly, since neither the top nor the bottom part of the functions domain includes 0, this means that 0 is not in the domain of the function at all. Therefore, $f(0) = $ undefined.

Example 2:

Evaluate the function $g(x) = \begin{cases} 2x - 3 & if \ x < -1 \\ 5 & if -1 \le x < 1 \\ 2x^2 & if \ x \ge 1 \end{cases}$ at the given values

  a. $g(-2)$ b. $g(0)$ c. $g(2)$ d. $g(-1)$ e. $g(1)$

Solution:

  a. As we did in example 1, we just need to put each value into it's proper piece.

  So, here, the value of -2 is in the top part of the function. So we get

  $$g(-2) = 2(-2) - 3 = -4 - 3 = -7$$

b. This time, 0 is in the middle section of the function. So we have

$$g(0) = 5$$

c. Here, the value is in the bottom part so we get

$$g(2) = 2(2)^2 = 8$$

d. This time, we are working with a value that is on the edge of two parts of the function. We just need to pay close attention to our inequality signs. So, since the middle part includes the value of -1 (since it contains a less than **or equal to** -1), we put this value into the middle part. This gives
$$g(-1) = 5$$

e. Lastly, as in part d, we make sure we put it into the correct part. Here, it needs to go into the bottom part (since it includes the value of 1 with the greater than **or equal to** 1). This gives us
$$g(1) = 2(1)^2 = 2$$

There are numerous piecewise defined functions, however, one in particular is of special note. It's called the greatest integer function.

Definition: **Greatest Integer Function**- the function defined by the following

$\llbracket x \rrbracket$= the greatest integer less than or equal to x

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Essentially, the greatest integer function just rounds every value down, regardless if its value in the tenths place. Occasionally, you will hear it referred to as the "step" function mostly because of the graph, which we will talk about later.

The greatest integer function is a unique function in that it has a domain that is all real numbers but its range is just the integers. It can be very useful when trying to find examples that don't fit the "norm."

Let's see it in action.

Example 3:

Evaluate the function $f(x) = \llbracket x - 2 \rrbracket$ at the given values
     a. $f(5)$ b. $f(2.6)$ c. $f(-1.2)$ d. $f(0.3)$

Solution:
     a. To evaluate 5 in the greatest integer function, we first need to plug it in. This gives

$$f(5) = \llbracket 5 - 2 \rrbracket = \llbracket 3 \rrbracket$$

Now, we just need to figure out what the value of the greatest integer that is less than or equal to 3 would be. Obviously, that value would be 3.

b. Again, we start by plugging in the given value.
$$f(2.6) = \llbracket 2.6 - 2 \rrbracket = \llbracket 0.6 \rrbracket$$

So, since 0 is the greatest integer that is smaller than 0.6, the answer is 0.

c. Plugging in gives
$$f(-1.2) = [\![-1.2 - 2]\!] = [\![-3.2]\!]$$

The greatest integer that is smaller than -3.2 is -4. Therefore, the answer is -4.

d. Finally, as before we get
$$f(0.3) = [\![0.3 - 2]\!] = [\![-1.7]\!]$$

The greatest integer smaller that -1.7 is -2.

Simply put, the greatest integer function will always take any value between two integers and round it down to the lower value regardless of the value of the tenths place and any value of an integer itself, is just the value of the integer.

Another interesting piecewise defined function is, oddly enough, the absolute value function.

Definition: **Absolute Value**- (algebraic version) is defined as follows

$$|x| = \begin{cases} x & for\ x \geq 0 \\ -x & for\ x < 0 \end{cases}$$

This definition comes in very handy when we want to use the absolute value in a more algebraic way. It is helpful when we get into a calculus course and beyond.

If you carefully look at this definition, you can see that the range does indeed match that of the geometric definition, i.e. only positive values.

Let's see how, exactly to break down an absolute value with this definition.

Example 4:

Rewrite without absolute values.
    a. $f(x) = |x - 2|$ b. $g(x) = |3x - 2|$ c. $h(x) = |2 + 5x|$

Solution:
    a. The way to rewrite the absolute value is by making it into a piecewise defined function. The idea of the definition is that the values of the absolute value always change at the point at which the absolute value would have been zero.

    So the first thing we need to do is to find the value where the absolute value would be zero. So we simply solve the equation
$$|x - 2| = 0$$
Clearly, $x = 2$.

    So, the absolute value is positive when the values of x are larger than 2 (since they would still make the inside of the absolute value positive, and they would be negative when the values are less than 2 (since they would make the inside negative, and a negative of a negative is a positive).

    Therefore, we get
$$f(x) = |x - 2| = \begin{cases} x - 2 & if\ x \geq 2 \\ -(x - 2) & if\ x < 2 \end{cases}$$

More simply put

$$f(x) = \begin{cases} x - 2 & if\ x \geq 2 \\ -x + 2 & if\ x < 2 \end{cases}$$

b. Again, we need to find the value of the zero of the absolute value. Solving gives

$$|3x - 2| = 0$$
$$3x - 2 = 0$$
$$x = \frac{2}{3}$$

So we get

$$g(x) = |3x - 2| = \begin{cases} 3x - 2 & if\ x \geq \frac{2}{3} \\ -(3x - 2) & if\ x < \frac{2}{3} \end{cases}$$

Therefore,

$$g(x) = \begin{cases} 3x - 2 & if\ x \geq \frac{2}{3} \\ -3x + 2 & if\ x < \frac{2}{3} \end{cases}$$

c. Lastly, we proceed as we did in parts a and b.

$$|2 + 5x| = 0$$
$$2 + 5x = 0$$
$$x = -\frac{2}{5}$$

So we get

$$h(x) = |2 + 5x| = \begin{cases} 2 + 5x & if\ x \geq -\frac{2}{5} \\ -(2 + 5x) & if\ x < -\frac{2}{5} \end{cases}$$

Therefore,

$$h(x) = \begin{cases} 2 + 5x & if\ x \geq -\frac{2}{5} \\ -2 - 5x & if\ x < -\frac{2}{5} \end{cases}$$

Finally, and very importantly, we need to be able to graph a piecewise defined function. The simple way to do this is to simply graph each piece up to its domain defined endpoints. Whether or not we include the point, depends on what type of inequality symbol is used in the function.

We will see with the following example.

Example 5:

Graph the following.

a. $f(x) = \begin{cases} 2x + 1 & if\ x > 1 \\ x^2 - 1 & if\ x \leq 1 \end{cases}$ b. $f(x) = \begin{cases} -x^2 + 6 & if\ x < -2 \\ |x| & if\ -2 \leq x < 1 \\ \sqrt{x} & if\ x > 1 \end{cases}$ c. $f(x) = [\![x]\!]$

Solution:
a. To graph a piecewise defined function is fairly straight forward. We simply need to graph each piece over its domain. The only real "tricky" parts are that, first, you need to make

sure you pay close attention to whether or not an endpoint is included. If an endpoint is included, we make a dot on the graph and if not we make an open dot on the graph.

Secondly, we need to make sure we plot the endpoints for each domain portion even on the parts where the endpoint is not included (that's why we use the open dot). This assures us that the graph will actually cover the entire domain for each piece.

So, let's start by graphing each part. First, on the top part, we only want the part of the graph of y=2x+1 that is greater than 1. So we get
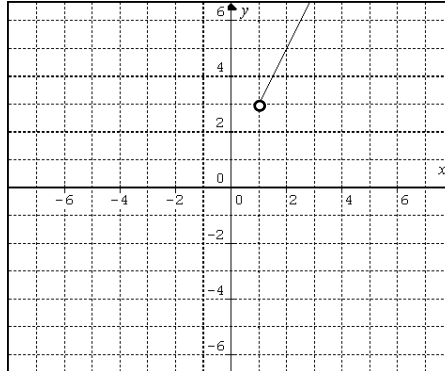
Notice that we included the point (1, 3) with an open dot because 1 is not included in the top part of the function, but we need to make sure that the graph has some sort of reference point as to where it begins. The open dot gives us that reference point.

Next we want just the part of $y = x^2 - 1$ that is less than or equal to 1. We get
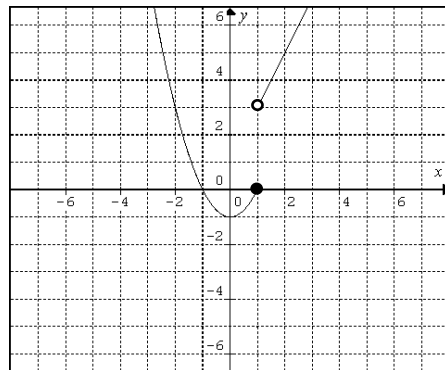
Again, note that we used is solid dot for the $y = x^2 - 1$ part. That is because the value of (1,0) is actually included in that part of the graph.

Since both parts are on the graph, our graph is complete.

b. This time since we have three parts to the piecewise defined function, we have to be even more careful with our endpoints. Just as we did in part a, we will graph each piece over its given domain.

Starting with the first (top) part we have
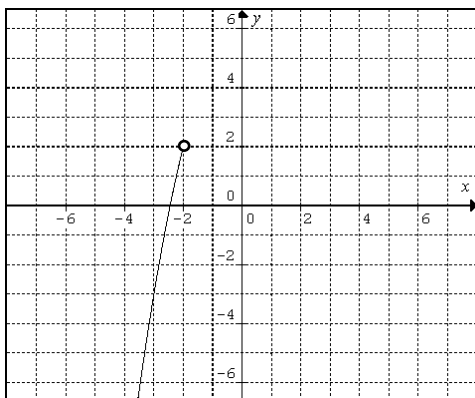
Next we add in the middle part. But first, notice the end point at -2 for both the top and middle parts give the same value, 2. This means we can just smoothly connect the pieces and forgo the "dots" since the ends connect together.

So we now have
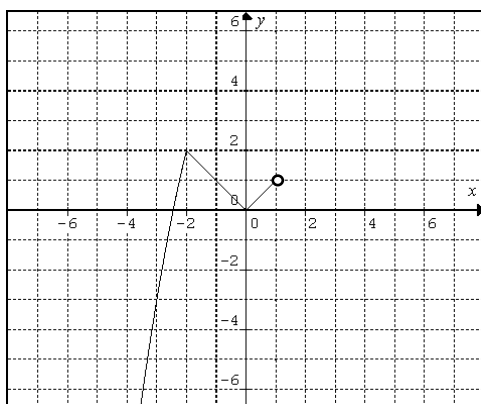
Finally, we graph the bottom part. Again, notice that the values on the endpoints agree. They both produce (1,1). However, since neither piece of the original function actually includes the value at x=1, we will have to keep it as an open dot in our graph.

Putting it together we get
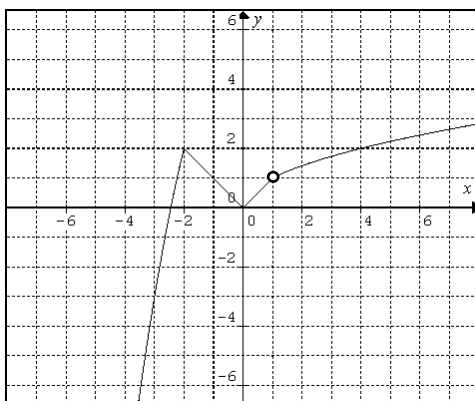
c. Lastly, to graph our greatest integer function, we need to approach things a little differently. Since we don't have the function given in terms of its pieces, we will need to use the definition and a little intuition to produce the graph.

First, notice that each value at an integer is the actual value. Also, we need to notice that all the values between 0 and 1 (for example) would always produce 0 (by definition, 0 is the greatest integer that is less than all values between 0 and 1). Likewise, all the values between 1 and 2 would give a 1, between 2 and 3 give a 2, etc.

So we can see we would get the following
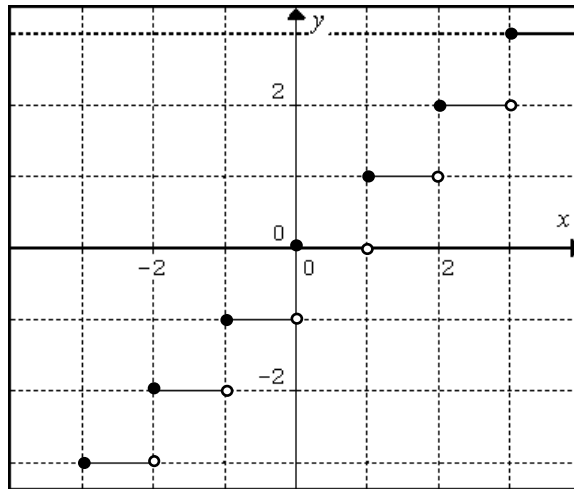
Now we can see why the greatest integer function is also called the "step" function. It's graph appears like steps up a staircase.

9.5 Exercises

1. Let $f(x) = \begin{cases} x & if\ x > 0 \\ 1 & if\ x \le 0 \end{cases}$. Evaluate the following
    a. $f(1)$
    b. $f(-1)$
    c. $f(-4)$
    d. $f(0)$

2. Let $f(x) = \begin{cases} 2x & if\ x > 1 \\ 3 & if\ x \le 1 \end{cases}$. Evaluate the following
    a. $f(2)$
    b. $f(-4)$
    c. $f(-1)$
    d. $f(1)$

3. Let $g(x) = \begin{cases} 0 & if\ x \ne 0 \\ 1 & if\ x = 0 \end{cases}$. Evaluate the following
    a. $g(1)$
    b. $g(-5)$
    c. $g(6)$
    d. $g(0)$

4. Let $g(x) = \begin{cases} 1 & if\ x = 1 \\ 2 & if\ x \ne 1 \end{cases}$. Evaluate the following
    a. $g(-1)$
    b. $g(2)$
    c. $g(5)$
    d. $g(1)$

5. Let $h(x) = \begin{cases} x^2 + 1 & if\ x \geq 2 \\ 2x - 3 & if\ x < 2 \end{cases}$. Evaluate the following
   a. $h(1)$
   b. $h(3)$
   c. $h(-2)$
   d. $h(2)$

6. Let $h(x) = \begin{cases} x - 5 & if\ x \leq -1 \\ x^3 - 2 & if\ x > -1 \end{cases}$.
   Evaluate the following
   a. $h(1)$
   b. $h(0)$
   c. $h(2)$
   d. $h(-1)$

7. Let $f(x) = \begin{cases} -1 & if\ x < 0 \\ 0 & if\ x = 0 \\ 1 & if\ x > 0 \end{cases}$. Evaluate the following
   a. $f(1)$
   b. $f(-1)$
   c. $f(2)$
   d. $f(0)$

8. Let $f(x) = \begin{cases} 2 & if\ x < 1 \\ 3 & if\ x = 1 \\ 4 & if\ x > 1 \end{cases}$. Evaluate the following
   a. $f(2)$
   b. $f(-3)$
   c. $f(0)$
   d. $f(1)$

9. Let $g(x) = \begin{cases} x & if\ x < -1 \\ 1 & if\ -1 \leq x < 2 \\ -x^2 & if\ x \geq 2 \end{cases}$.
   Evaluate the following
   a. $g(0)$
   b. $g(-3)$
   c. $g(-1)$
   d. $g(2)$

10. Let $g(x) = \begin{cases} 3x & if\ x \leq -2 \\ 2 & if\ -2 < x < 2 \\ x^3 - 1 & if\ x \geq 2 \end{cases}$.
    Evaluate the following
    a. $g(0)$
    b. $g(-3)$
    c. $g(-2)$
    d. $g(2)$

11. Let $h(x) = \begin{cases} 2x - 1 & if\ x < -2 \\ x^2 - x & if\ -2 < x < 2 \\ -2x + 1 & if\ x > 2 \end{cases}$.
    Evaluate the following
    a. $h(3)$
    b. $h(-3)$
    c. $h(0)$
    d. $h(2)$

12. Let $h(x) = \begin{cases} 1 - x^2 & if\ x < 0 \\ 1 - x & if\ 0 < x < 1 \\ 1 & if\ x > 1 \end{cases}$.
    Evaluate the following
    a. $h(0.5)$
    b. $h(-2)$
    c. $h(2)$
    d. $h(0)$

13. Let $f(x) = [\![x]\!]$. Evaluate the following
    a. $f(1.2)$
    b. $f(-3.8)$
    c. $f(3)$
    d. $f(a - 1)$

14. Let $f(x) = [\![x]\!] + 1$. Evaluate the following
    a. $f(2.5)$
    b. $f(-1.7)$
    c. $f(-2)$
    d. $f(n + m)$

15. Let $f(x) = [\![x - 1]\!]$. Evaluate the following
    a. $f(1.8)$
    b. $f(-2.7)$
    c. $f(1)$
    d. $f(x + h)$

16. Let $f(x) = [\![x + 2]\!]$. Evaluate the following
     a. $f(-3.2)$
     b. $f(1.3)$
     c. $f(-2)$
     d. $f(a + 2)$

Rewrite without absolute values.
17. $f(x) = |x + 1|$
18. $f(x) = |x - 1|$
19. $f(x) = |2x|$
20. $f(x) = |3x|$

21. $f(x) = |2x - 1|$
22. $f(x) = |3x + 2|$
23. $f(x) = |1 - 4x|$
24. $f(x) = |2 - 3x|$

Graph.

25. $f(x) = \begin{cases} x + 1 & if \ x > 0 \\ 1 & if \ x \le 0 \end{cases}$

26. $f(x) = \begin{cases} 2x & if \ x > 1 \\ 3 & if \ x \le 1 \end{cases}$

27. $g(x) = \begin{cases} 0 & if \ x \ne 0 \\ 1 & if \ x = 0 \end{cases}$

28. $g(x) = \begin{cases} 1 & if \ x = 1 \\ 2 & if \ x \ne 1 \end{cases}$

29. $h(x) = \begin{cases} x^2 + 1 & if \ x \ge 2 \\ 2x - 3 & if \ x < 2 \end{cases}$

30. $h(x) = \begin{cases} x - 5 & if \ x \le -1 \\ x^3 - 2 & if \ x > -1 \end{cases}$

31. $f(x) = \begin{cases} x^3 & if \ x \le 0 \\ |x| & if \ x > 0 \end{cases}$

32. $f(x) = \begin{cases} -x^2 + 1 & if \ x < -1 \\ \sqrt{x + 1} & if \ x \ge -1 \end{cases}$

33. $f(x) = \begin{cases} -1 & if \ x < 0 \\ 0 & if \ x = 0 \\ 1 & if \ x > 0 \end{cases}$

34. $f(x) = \begin{cases} 2 & if \ x < 1 \\ 3 & if \ x = 1 \\ 4 & if \ x > 1 \end{cases}$

35. $g(x) = \begin{cases} x & if \ x < -1 \\ 1 & if \ -1 \le x < 2 \\ -x^2 & if \ x \ge 2 \end{cases}$

36. $g(x) = \begin{cases} 3x & if \ x \le -2 \\ 2 & if \ -2 < x < 2 \\ x^3 - 1 & if \ x \ge 2 \end{cases}$

37. $h(x) = \begin{cases} 2x - 1 & if \ x \le -2 \\ -x^2 - 1 & if \ -2 < x < 2 \\ -2x - 1 & if \ x \ge 2 \end{cases}$

38. $h(x) = \begin{cases} 1 - x^2 & if \ x < 0 \\ 1 - x & if \ 0 < x < 1 \\ 1 & if \ x > 1 \end{cases}$

39. $f(x) = \begin{cases} 2x & if \ x < 0 \\ x^2 & if \ 0 < x < 1 \\ \sqrt{x - 1} + 1 & if \ 1 < x < 2 \\ -x + 4 & if \ x > 2 \end{cases}$

40. $f(x) = \begin{cases} 3x - 1 & if \ x \le 0 \\ |x - 2| & if \ 0 < x \le 2 \\ -x^2 & if \ 2 < x < 3 \\ 5 & if \ x > 3 \end{cases}$

41. $f(x) = [\![x - 1]\!]$

42. $f(x) = [\![x + 1]\!]$

43. $f(x) = [\![x]\!] + 2$

44. $f(x) = [\![x]\!] - 1$

45. $f(x) = [\![x + 2]\!] - 2$

46. $f(x) = [\![x - 3]\!] + 3$