9.5 The Algebra of Functions

In this section, we will take a look at all our basic operations as they apply to functions.

Operations on Functions	
If f and g are functions and x is and element of the domain of each function then,	
$(f+g)(x) = f(x) + g(x)$	$(fg)(x) = f(x)g(x)$
$(f-g)(x) = f(x) - g(x)$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

This just means in order to perform a basic operation on two functions, you simply need to do that particular operation pointwise.

Example 1:

Let $f(x) = x^2 - x$ and $g(x) = 3x - 2$. Find the following.

a. $(f-g)(3)$ b. $(f+g)(2)$ c. $(fg)(-1)$ d. $(f+g)(x)$

e. $(fg)(x)$ f. $\left(\frac{f}{g}\right)(0)$ g. $\left(\frac{f}{g}\right)\left(\frac{2}{3}\right)$

Solution:

a. First we recall the rule given above $(f-g)(x) = f(x) - g(x)$.
There are two primary ways in which we can approach this problem. One way is to just replace the x in the rule with 3 and simplify as follows:

$$\begin{aligned}(f-g)(3) &= f(3) - g(3)\\ &= (3^2) - 3 - (3(3) - 2)\\ &= 9 - 3 - 9 + 2\\ &= -1\end{aligned}$$

The alternate way to do this problem is to just find $(f-g)(x)$ in general and then substitute the value in the end.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= x^2 - x - (3x - 2)\\ &= x^2 - 4x + 2\end{aligned}$$

$$(f-g)(3) = 3^2 - 4(3) + 2 = -1$$

We can see they both give us –1. We prefer the latter method since once the expression for $(f-g)(x)$ has been found we can evaluate it at as many values as we would like.

b. Again we will start by finding $(f+g)(x)$ and then evaluate at 2.

$$(f+g)(x) = f(x) + g(x)$$
$$= x^2 - x + (3x - 2)$$
$$= x^2 + 2x - 2$$
$$(f+g)(2) = 2^2 + 2(2) - 2 = -2$$

c.

$$(fg)(x) = f(x)g(x)$$
$$= (x^2 - x)(3x - 2)$$
$$= 3x^3 - 5x^2 + 2x$$
$$(fg)(-1) = 3(-1)^3 - 5(-1)^2 + 2(-1)$$
$$= -10$$

c. Since we have already found $(f+g)(x)$ for part b. we simply need to refer back.

$$(f+g)(x) = x^2 + 2x - 2$$

d. We have also already found $(fg)(x) = 3x^3 - 5x^2 + 2x$ in part c above.

e. We will again start by finding $\left(\frac{f}{g}\right)(x)$, then evaluate at 0.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$= \frac{x^2 - x}{3x - 2}$$
$$\left(\frac{f}{g}\right)(0) = \frac{0^2 - 0}{3(0) - 2}$$
$$= \frac{0}{-2}$$
$$= 0$$

f. In part e. we found $\left(\frac{f}{g}\right)(x) = \frac{x^2 - x}{3x - 2}$. Evaluating at $\frac{2}{3}$ we get

$$\left(\frac{f}{g}\right)\left(\frac{2}{3}\right) = \frac{\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)}{3\left(\frac{2}{3}\right) - 2}$$
$$= \frac{\frac{4}{9} - \frac{2}{3}}{0}$$

But we recall that we cannot have a zero in the denominator. Also, from the rules above, we know that $g(x) \neq 0$. Thus, we can see that $\frac{2}{3}$ is not in the domain of $\left(\frac{f}{g}\right)(x)$.

Thus, we can say $\left(\frac{f}{g}\right)\left(\frac{2}{3}\right)$ is not a real number.

We now turn our attention to another very important operation on functions, the composition.

Definition: Composition of two functions
Let f and g be two functions such that $g(x)$ is in the domain of f for all x in the domain of g. Then the composition of f and g, written $f \circ g$, is the function given by $(f \circ g)(x) = f(g(x))$.

It usually helps to visualize the composition as $f\left({\scriptstyle g(x)} \right)$.

The composition idea is one of putting an entire function into another function.

<u>Example 2</u>:

Let $f(x) = 2x^2 - x + 1$, $g(x) = 2x - 3$ and $h(x) = \dfrac{1}{x+1}$. Find the following.

a. $(f \circ g)(0)$ b. $(h \circ g)(2)$ c. $(h \circ f)(2)$ d. $(f \circ g)(x)$
e. $(g \circ h)(x)$ f. $(f \circ f)(x)$

Solution:

a. First, by the definition we know $(f \circ g)(x) = f(g(x))$. So we really want $f(g(0))$. Like the example above, we can do this by finding the formula for $(f \circ g)(x)$ first and then evaluate it at 0. So, if we insert $g(x) = 2x - 3$ into $f(g(x))$ and simplify we get

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x-3) \\ &= 2(2x-3)^2 - (2x-3) + 1 \\ &= 2(4x^2 - 12x + 9) - 2x + 3 + 1 \\ &= 8x^2 - 24x + 18 - 2x + 4 \\ &= 8x^2 - 24x + 22\end{aligned}$$

Now we evaluate it for x=0

$$\begin{aligned}(f \circ g)(0) &= 8(0)^2 - 24(0) + 22 \\ &= 22\end{aligned}$$

b. In a similar fashion we start by finding $(h \circ g)(x)$, the evaluate as follows

$$
\begin{aligned}
(h \circ g)(x) &= h(g(x)) \\
&= h(2x-3) \\
&= \frac{1}{(2x-3)+1} \\
&= \frac{1}{2x-2} \\
(h \circ g)(2) &= \frac{1}{2(2)-2} \\
&= \frac{1}{4}
\end{aligned}
$$

c.

$$
\begin{aligned}
(h \circ f)(x) &= h(f(x)) \\
&= h(2x^2 - x + 1) \\
&= \frac{1}{(2x^2 - x + 1) + 1} \\
&= \frac{1}{2x^2 - x + 2} \\
(h \circ f)(2) &= \frac{1}{2(2)^2 - (2) + 2} \\
&= \frac{1}{8}
\end{aligned}
$$

d. In part a. above we already found $(f \circ g)(x) = 8x^2 - 24x + 22$.

e.

$$
\begin{aligned}
(g \circ h)(x) &= g(h(x)) \\
&= g\left(\frac{1}{x+1}\right) \\
&= 2\left(\frac{1}{x+1}\right) - 3 \\
&= \frac{2}{x+1} - 3
\end{aligned}
$$

e. Lastly, we want to take the composition of a function with itself. We do this in the same manner as all other compositions. We simply insert f into f and simplify.

$$
\begin{aligned}
(f \circ f)(x) &= f(f(x)) \\
&= f\left(2x^2 - x + 1\right) \\
&= 2\left(2x^2 - x + 1\right)^2 - \left(2x^2 - x + 1\right) + 1 \\
&= 2\left(4x^4 - 4x^3 + 5x^2 - 2x + 1\right) - 2x^2 + x - 1 + 1 \\
&= 8x^4 - 8x^3 + 10x^2 - 4x + 2 - 2x^2 + x \\
&= 8x^4 - 8x^3 + 8x^2 - 3x + 2
\end{aligned}
$$

9.5 Exercises

Let $f(x) = 2x - 1$ and $g(x) = x^2 - 2x + 1$. Find the following.

1. $(f+g)(-1)$
2. $(f-g)(2)$
3. $(g-f)(x)$
4. $(fg)(-2)$
5. $\left(\frac{f}{g}\right)(0)$
6. $\left(\frac{g}{f}\right)(x)$
7. $(f-g)(a+h)$
8. $(f+g)(2+h)$
9. $\left(\frac{f}{g}\right)(ab)$
10. $(fg)(x)$

Let $g(x) = \frac{1}{x+1}$ and $h(x) = \frac{x}{x+1}$. Find the following.

11. $(g+h)(1)$
12. $(g-h)(-2)$
13. $(gh)(x)$
14. $\left(\frac{g}{h}\right)(-2)$
15. $(h-g)(x)$
16. $(g+h)\left(x^2\right)$
17. $(h-g)(a-1)$
18. $(gh)(0)$
19. $\left(\frac{h}{g}\right)(x)$
20. $(h-g)(a+b)$

Let $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 1$ and $h(x) = \frac{2}{x}$. Find the following.

21. $(f+g)(x)$
22. $(g-g)(x)$
23. $(h-g)(x)$
24. $(g-h)(x)$
25. $(gh)(x)$
26. $(fh)(x)$
27. $(fg)(x)$
28. $\left(\frac{g}{f}\right)(x)$
29. $\left(\frac{g}{h}\right)(x)$
30. $\left(\frac{f}{h}\right)(x)$

Let $f(x) = 3x + 1$ and $g(x) = 2x^2 - x + 4$. Find the following.

31. $(f \circ g)(-1)$
32. $(f \circ g)(0)$
33. $(g \circ f)(2)$
34. $(g \circ f)(-3)$
35. $(f \circ g)(-4)$
36. $(g \circ f)(6)$
37. $(f \circ g)(x)$
38. $(g \circ f)(x)$
39. $(f \circ f)(x)$
40. $(g \circ g)(x)$

Let $g(x)=\dfrac{1}{x+1}$ and $h(x)=\dfrac{2}{x}$. Find the following.

41. $(g\circ h)(2)$
42. $(h\circ g)(-1)$
43. $(h\circ g)(0)$
44. $(g\circ h)(0)$
45. $(h\circ h)(2)$
46. $(g\circ g)(0)$
47. $(g\circ h)(x)$
48. $(h\circ g)(x)$
49. $(h\circ h)(x)$
50. $(g\circ g)(x)$

Let $f(x)=\sqrt{x+1}$, $g(x)=x^2-2$ and $h(x)=|2x-1|$. Find the following.

51. $(f\circ g)(2)$
52. $(g\circ f)(0)$
53. $(h\circ f)(3)$
54. $(g\circ h)(-1)$
55. $(f\circ h)(t)$
56. $(g\circ f)(c)$
57. $(g\circ f)(8+a)$
58. $(f\circ g)(\Delta)$
59. $(g\circ g)(a-b)$
60. $(h\circ g)(a+b)$
61. $(g\circ f)(x+h)$
62. $(h\circ f)(x)$
63. $(g\circ f)(x)$
64. $(h\circ g)(x)$
65. $(g\circ h)(x)$
66. $(f\circ h)(x)$
67. $(f\circ g)(x)$
68. $(h\circ h)(x)$
69. $(g\circ g)(x)$
70. $(f\circ f)(x)$