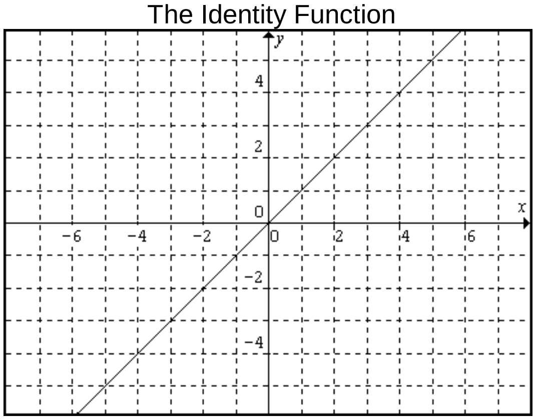
# 9.4 Graphing by Shifting, Reflecting and Dilation

In the previous section we talked about graphing functions. We saw that the graph of $f(x)=(x-2)^2$ is just the basic graph $f(x)=x^2$ "moved" over to the right two units. Similarly, $f(x)=-x^2+2$ is just the basic graph $f(x)=x^2$ "flipped over" and "moved" up two units. In this section we want to rigorously define what we mean by "moved" and "flipped over." These ideas are called shifting (or translating) and reflecting, respectively.

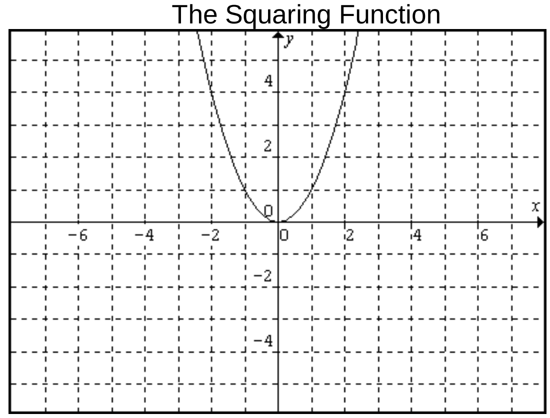
Before we define translating and reflecting mathematically we need to know the graphs of six basic functions. These functions are $y=x$, $y=x^2$, $y=x^3$, $y=\sqrt{x}$, $y=|x|$ and $y=\frac{1}{x}$. In section 9.3 we learned that we can find the graph of these functions by evaluating them at several points. We will leave it to the reader to verify the graphs of the six basic functions given below.

The Identity Function



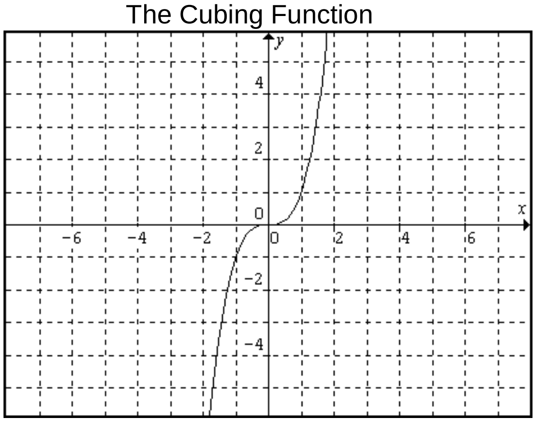

$y=x$

The Squaring Function



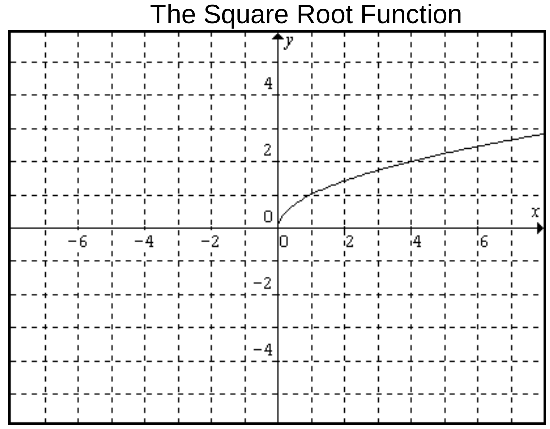

$y=x^2$

The Cubing Function



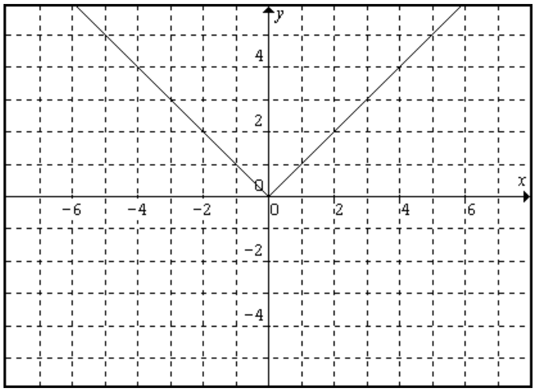

$y=x^3$

The Square Root Function



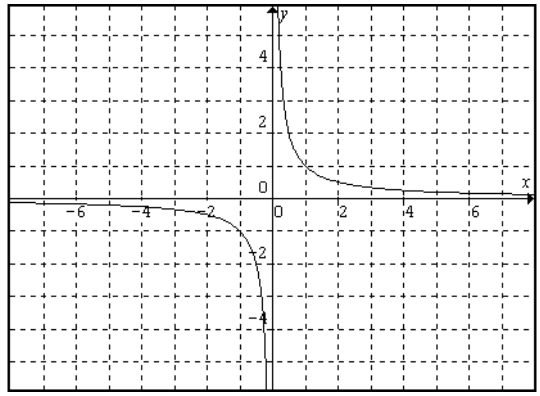

$y=\sqrt{x}$

The Absolute Value Function




$$y = |x|$$

The Reciprocal Function



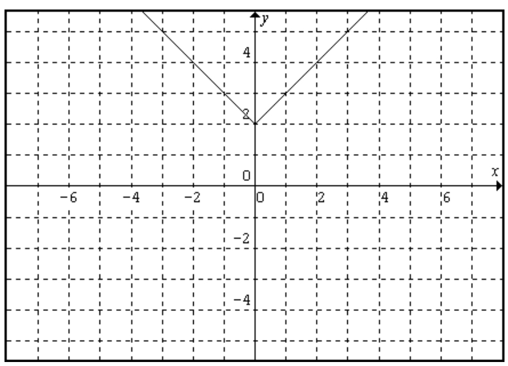

$$y = \frac{1}{x}$$

Now that we know the graphs of these six basic functions we can turn our attention to translating and reflecting.

### *Translations of the basic graph*

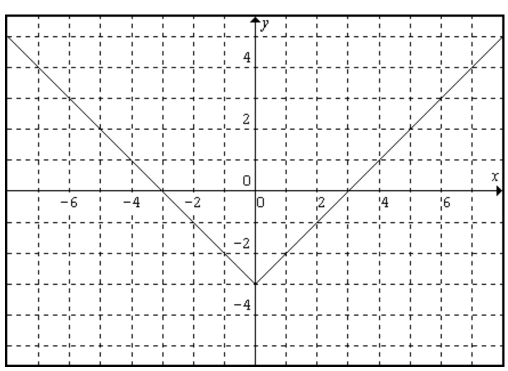
There are two types of translations that we would like to investigate, the vertical translation and the horizontal translation.

Consider the function $f(x) = |x| + 2$. By our evaluating we get the graph




Notice that this is just the graph of $f(x) = |x|$ moved up 2 units.

Likewise the graph of $f(x) = |x| - 3$ is

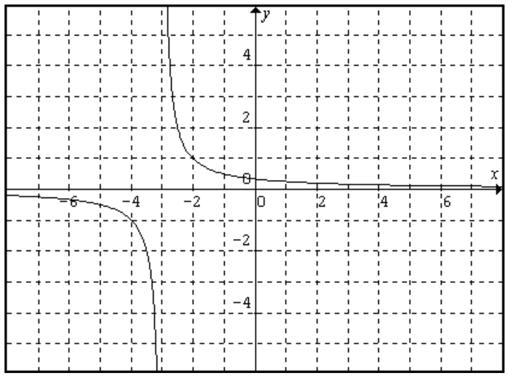


Again this is just the graph of $f(x)=|x|$ moved down 3 units.

These are known as vertical translations.

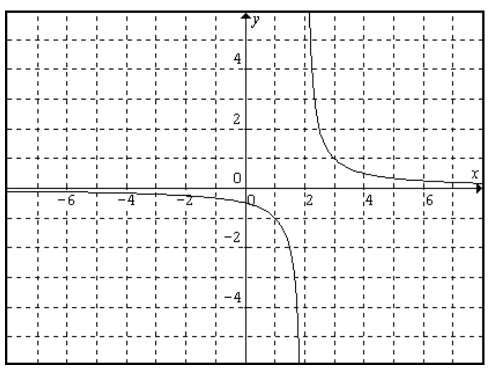
| **Vertical Translations** |
|---|
| The graph of $y=f(x)+k$ is the graph of $y=f(x)$ shifted $k$ units upward and $y=f(x)-k$ is the graph of $y=f(x)$ shifted $k$ units downward. |

Now lets consider the function $f(x)=\dfrac{1}{x+3}$. Again by evaluating we get the graph




Notice that this is just the graph of $f(x)=\dfrac{1}{x}$ moved to the left 3 units.

Similarly the graph of the function $f(x)=\dfrac{1}{x-2}$ is




Again notice that this is just the graph of $f(x)=\dfrac{1}{x}$ moved to the right 2 units.

These are known as horizontal translations.

| **Horizontal Translations** |
|---|
| The graph of $y=f(x-h)$ is the graph of $y=f(x)$ shifted $h$ units to the right and $y=f(x+h)$ is the graph of $y=f(x)$ shifted $h$ units to the left. |

(Note that the direction of shifting is the non-natural direction with horizontal translations)

One way to remember the horizontal and vertical translations is if you are changing the function by adding or subtracting <u>outside</u> then it is a vertical translation, and if it is on the <u>inside</u> then it is a horizontal translation.

Example 1

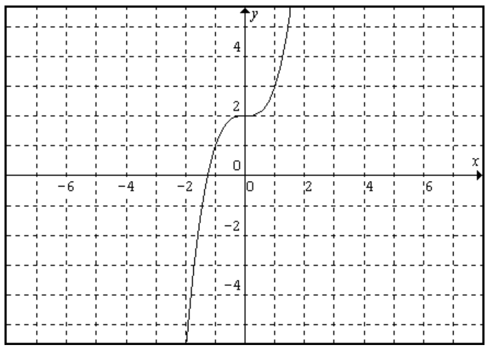
Graph the following and identify the translations.

a. $f(x) = x^3 + 2$

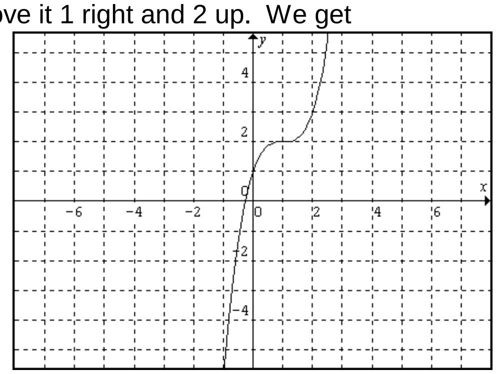
b. $g(x) = (x-1)^3 + 2$

*Solution:*

a. First we note what the basic function is. In this case we are dealing with the function $y = x^3$. But the +2 on the outside of the function so it is a vertical translation. From above, we have a translation of 2 units upward. So we start with the basic function and move it 2 units up. We get



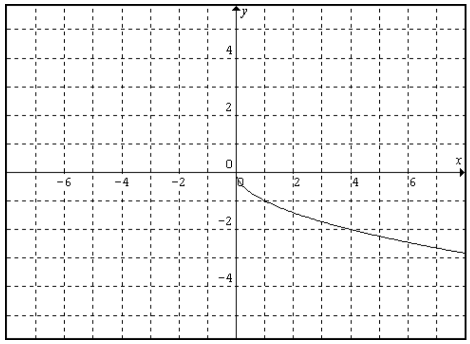

b. Again make a note of the basic function. Again it is $y = x^3$. However this time we have two things going on. We have a –1 on the inside and a +2 on the outside. These correspond to a horizontal translation right 1 unit and a vertical translation upward 2 units. So start with the basic function and move it 1 right and 2 up. We get




## *Reflections of the basic graph*

There are two types of reflections that want to consider, the x-axis reflection and the y-axis reflection.

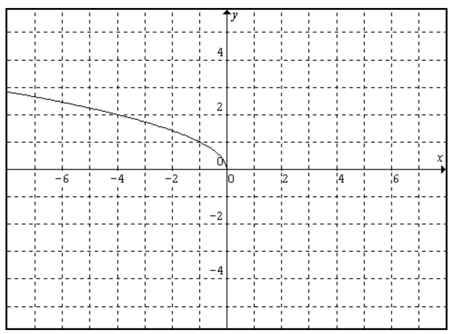
Consider the function $f(x) = -\sqrt{x}$. By evaluating we get the graph




Notice that this is just the graph of $f(x) = \sqrt{x}$ flipped upside down.
This is known as a <u>reflection across the x-axis</u>.

| **x-Axis Reflection** |
|---|
| The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected across the x-axis. |

Similarly the graph of $f(x) = \sqrt{-x}$ is




Again notice that this is just the graph of $f(x) = \sqrt{x}$ flipped over to the left.
This is known as a <u>reflection across the y-axis</u>.

| **y-Axis Reflection** |
|---|
| The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. |

It is sometimes helpful to remember that if the negative is on the <u>outside</u> it is a reflection across the x-axis, and if it is on the <u>inside</u> it is a reflection across the y-axis.

## *Dilation, a.k.a. Vertical Stretching and Shrinking*

The translations and reflections are called <u>rigid transformations</u> because they do not change the basic shape of the graph, they mearly change the position of the graph. However, there are two types of transformations that do change the basic shape of the graph. We can make a graph "narrower" or "wider." Mathematically we call this <u>dilation</u>.

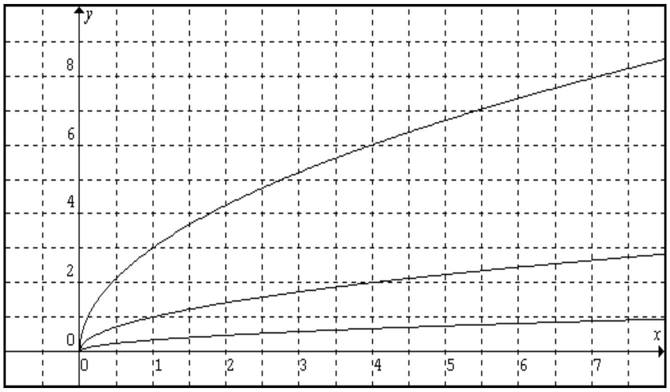
| **Dilation** |
|---|
| The graph of $y = cf(x)$ is the graph of $y = f(x)$ shrunk by a factor of *c* if $0 < c < 1$, and stretched by a factor of *c* if $c > 1$. In either case, the graph is formed by multiplying the y coordinates by *c.* |

Example 2

Graph $f(x)=3\sqrt{x}$ and $f(x)=\frac{1}{3}\sqrt{x}$ .

*Solution:*

From the box above we need to multiply the y coordinates of $y=\sqrt{x}$ by 3 and by $\frac{1}{3}$ respectively to obtain each graph. Below we have graphed each of these functions.




Lets put it all together for our last example.

Example 3

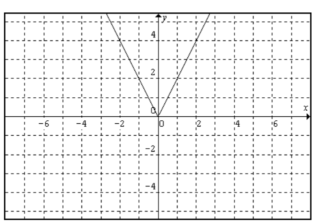
Graph $g(x)=-2|x+1|-3$. Identify all the transformations.

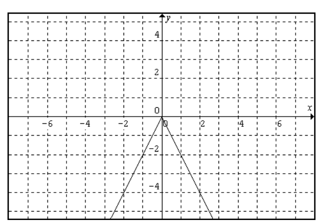
*Solution:*

This is the basic function $y=|x|$. From everything that we did above we know that this graph is different from the basic graph by the following transformations: Stretched by a factor of 2, reflected across the x-axis, shifted left 1 unit and shifted down 3 units

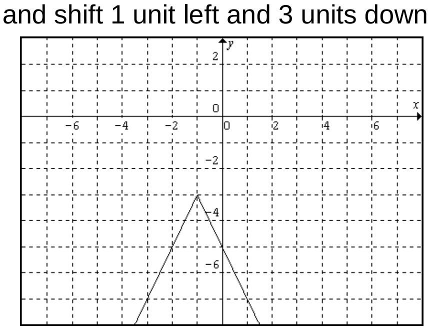
We should always take care of the stretching or shrinking and reflections first.

Start by graphing $y=2|x|$ by plotting at least 3 points. Then reflect that graph over the x-axis.





Finally, take the last graph and shift 1 unit left and 3 units down to complete the graph.



## 9.4 Exercises

Graph each of the following using transformations. Identify the translations and reflections.

1. $f(x)=|x|+3$
2. $f(x)=|x|-2$
3. $f(x)=-|x|$
4. $h(x)=3|x|$
5. $h(x)=\sqrt{x+3}$
6. $h(x)=\sqrt{x}+4$
7. $g(x)=-\sqrt{x}-2$
8. $g(x)=-\sqrt{x-5}$
9. $g(x)=|-x|$
10. $f(x)=\dfrac{1}{x}-4$
11. $h(x)=\dfrac{1}{x+1}$
12. $g(x)=\dfrac{1}{x}+1$
13. $f(x)=\dfrac{3}{x-4}$
14. $f(x)=-4x^3$
15. $f(x)=-2x^3-2$
16. $h(x)=(x-2)^2-4$
17. $h(x)=(x+3)^3+1$
18. $h(x)=-|x+4|-1$
19. $g(x)=-\dfrac{1}{x-2}+3$
20. $g(x)=\sqrt{x-1}+4$
21. $g(x)=-\sqrt{-(x-2)}+1$
22. $f(x)=-3|x+1|-2$
23. $h(x)=-\frac{1}{2}(x+2)^2-3$
24. $g(x)=(-x)^3+2$
25. $f(x)=\dfrac{1}{x-2}+3$
26. $f(x)=\dfrac{1}{x+3}-2$
27. $f(x)=-\dfrac{2}{x}-4$
28. $f(t)=2(t-2)^3+3$
29. $g(u)=-2(u+3)^3-4$
30. $h(t)=-4\sqrt{t+1}-2$
31. $g(x)=3-2|x+1|$
32. $f(x)=\frac{1}{3}(x-3)^2+1$
33. $h(x)=1-\dfrac{1}{2(x-3)}$
34. $f(x)=2-\sqrt{-(x-2)}$
35. $f(u)=(u-\frac{1}{2})^3-1$
36. $f(x)=-2|x-2|-2$
37. $g(x)=(x+1)^2+\frac{1}{4}$
38. $h(x)=\frac{1}{3}-\frac{1}{4}(x+\frac{1}{2})^3$
39. $g(x)=-\frac{1}{3}\sqrt{x-\frac{1}{3}}-\frac{2}{3}$
40. $f(x)=\dfrac{2}{3(x-2)}+\dfrac{1}{4}$
41. $g(x)=\dfrac{1}{2}-\dfrac{1}{2(x+1)}$
42. $h(x)=-\frac{1}{5}|x+\frac{1}{4}|-\frac{1}{3}$
43. Let $y=f(x)$ be the function defined by the line segment connecting the points $(1,2)$ and $(3,1)$. Graph each of the following transformations of $y=f(x)$.
    $a.\ \ y=f(x)-2 \quad b.\ \ y=f(x+1) \quad c.\ \ y=-f(x) \quad d.\ \ y=f(x-1)+2$

44. Let $y = f(x)$ be the function defined by the line segment connecting the points $(-1,4)$ and $(2,5)$. Graph each of the following transformations of $y = f(x)$.

$a.\ \ y = f(x)+1 \quad b.\ \ y = f(x+2) \quad c.\ \ y = f(-x) \quad d.\ \ y = -f(x+3)-2$

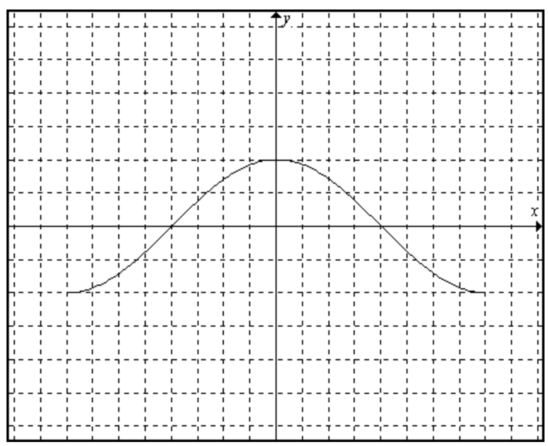
45. Let $y = f(x)$ be the function defined by the line segment connecting the points $(2,-2)$ and $(6,-1)$. Graph each of the following transformations of $y = f(x)$.

$a.\ \ y = f(x-3) \quad b.\ \ y = f(x)+4 \quad c.\ \ y = -f(-x) \quad d.\ \ y = -f(x+1)-2$

46. Let $y = f(x)$ be the function defined by the line segment connecting the points $(4,1)$ and $(-1,-1)$. Graph each of the following transformations of $y = f(x)$.

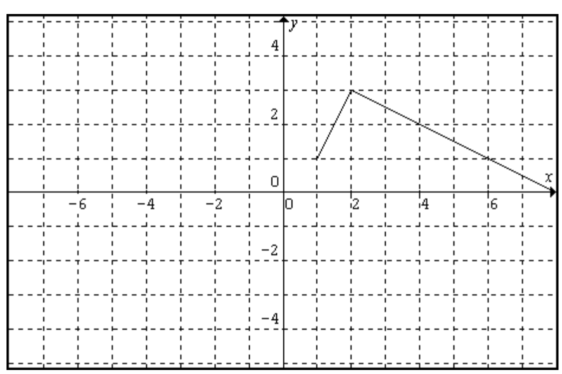
$a.\ \ y = -f(x)+3 \quad b.\ \ y = f(x+3) \quad c.\ \ y = f(-(x-1)) \quad d.\ \ y = -f(x-1)-2$

47. The graph of $y = f(x)$ with a range of $[-1, 1]$ is shown below. Sketch the graph of each of the following transformations of $y = f(x)$.




$a.\ \ y = f(x)+3 \quad b.\ \ y = f(x-2) \quad c.\ \ y = -f(x) \quad d.\ \ y = f(x+3)-2$

48. The graph of $y = f(x)$ is shown below. Sketch the graph of each of the following transformations of $y = f(x)$.




$a.\ \ y = f(x)+3 \quad b.\ \ y = f(x-2) \quad c.\ \ y = -f(x) \quad d.\ \ y = f(x+3)-2$

Sometimes a function is not written in the proper form and we must do some initial work to put it in the correct form before determining the transformations. For example, $f(x)=\sqrt{1-x}$ is not in the correct form but if we factor out a negative we get $f(x)=\sqrt{-(x-1)}$ which is easy to graph. Use factoring to write the function in the correct form. Then graph.

49. $f(x)=|2-x|$

50. $f(x)=\sqrt{3-x}$

51. $g(x)=\dfrac{1}{1-x}$

52. $g(x)=(4-x)^3$

53. $h(x)=1-\sqrt{1-x}$

54. $h(x)=|5-x|+1$

55. $f(x)=2\sqrt{1-x}+1$

56. $h(x)=3+\sqrt{4-x}$

57. $f(x)=\frac{1}{2}(2-x)^2-1$

58. $g(x)=\dfrac{2}{2-x}-2$

59. $f(x)=2-3(3-x)^3$

60. $g(x)=\frac{2}{3}(2-x)^2-4$