9.3 Graphing Functions by Plotting Points, The Domain and Range of Functions

Now that we have a basic idea of what functions are and how to deal with them, we would like to start talking about the graph of a function.

First, the following definition,

> **Definition:** The graph of a function is the set of all points of the form $(x, f(x))$, where the x values are in the domain of f.

So, the most fundamental way in which we can graph a function is the same as the most fundamental way we can graph anything, that is, by plotting points.

Before we get to that, recall the following:

> Domain of a function = set of all first coordinates, Range of a function = set of all second coordinates

So that means the domain is the set of x-values that can be graphed, and the range is the f(x) or y-values that get graphed.

Using these definitions and concepts we can begin to graph functions and determine their domains and ranges.

Example 1:

Graph $f(x)=\sqrt{1+x}$. Determine the domain and range.

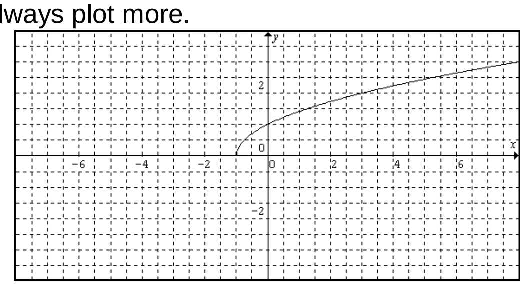
Solution:

First we need to construct a table of values. We can choose any values we would like that will return a real number. Since our function has a radical we will have to be very careful how we choose our values. It should be clear that if we take any x value smaller than –1 we will end up with a negative under the radical. Which is not a real number. So lets start at –1 and take larger values.

X	f(x)
-1	0
0	1
1	1.4
2	1.7
3	2

$$f(-1)=\sqrt{1+(-1)}=\sqrt{0}=0$$
$$f(0)=\sqrt{1+(0)}=\sqrt{1}=1$$
$$f(1)=\sqrt{1+(1)}=\sqrt{2}\approx 1.4$$
$$f(2)=\sqrt{1+(2)}=\sqrt{3}\approx 1.7$$
$$f(3)=\sqrt{1+(3)}=\sqrt{4}=2$$

We can now plot these points and graph the curve. If this is not enough points to tell what the graph looks like, we can always plot more.



Now, using the graph we can determine the domain and range of the function. From above, we know that the domain is the x-values. So by the graph, we can see that the only x-values that have a y-value associated with them are from –1 on. So as an interval we write that as $[-1,\infty)$. Likewise for the range we look at the y-values that get used. We can see that all the y-values from 0 upwards would get used. We write this in interval notation as $[0,\infty)$.

Example 2:

Graph $f(x)=|2x-1|$. Determine the domain and range.

Solution:

Again we will plot several points to get an idea what the graph of the function is doing. If its not enough we can always plot more. We usually like to start with easy values.

X	f(x)
-2	5
-1	3
0	1
1	1
2	3

$f(-2)=|2(-2)-1|=|-5|=5$

$f(-1)=|2(-1)-1|=|-3|=3$

$f(0)=|2(0)-1|=|-1|=1$

$f(1)=|2(1)-1|=|1|=1$

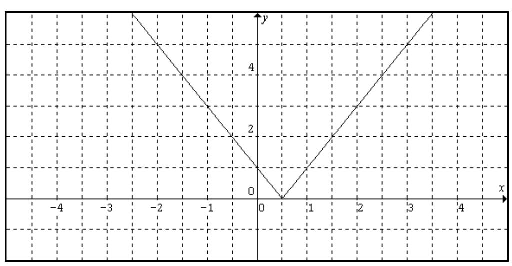
$f(2)=|2(2)-1|=|3|=3$

If we plot these points we find that we cannot tell what happens on the graph between 0 and 1. So lets plot another point say $\frac{1}{2}$.

X	f(x)
$\frac{1}{2}$	0

$f\left(\frac{1}{2}\right)=\left|2\left(\frac{1}{2}\right)-1\right|=|0|=0$

So plotting these points we get the following



In general, graphs of functions that contain an absolute value will have at least one sharp corner like the one seen above.

Again the domain of the function is the x-values, we can clearly see all x-values will get used. As an interval we write $(-\infty,\infty)$. For the range, we look at the used y-values and we can see that we have from 0 up. That is, $[0,\infty)$.

Example 3:

Graph $f(x)=x^3-x^2-x-1$. Determine the domain and range.

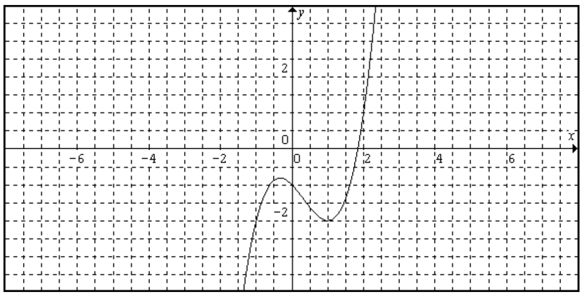
Solution:

Again we will plot several points.

X	f(x)
-2	-11
-1	-2
0	-1
1	-2
2	1

$$f(-2)=(-2)^3-(-2)^2-(-2)-1=-8-4+2-1=-11$$
$$f(-1)=(-1)^3-(-1)^2-(-1)-1=-1-1+1-1=-2$$
$$f(0)=(0)^3-(0)^2-(0)-1=0-0-0-1=-1$$
$$f(1)=(1)^3-(1)^2-(1)-1=1-1-1-1=-2$$
$$f(2)=(2)^3-(2)^2-(2)-1=8-4-2-1=1$$

Plotting these points we get.



Finally, we can see that the all x- and y-values will be used in this function. Thus, the domain and ranges are both $(-\infty,\infty)$.

As it turns out not every graph represents a function. Since, in order to have a function, one value of x into the function must give us only one f(x) value out of the function we get the following test for functions.

The Vertical Line Test

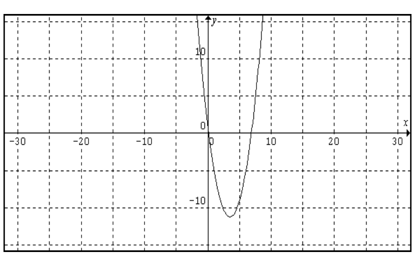
A graph defines a function if any vertical line intersects the graph at no more than one point.

What that means is in order to see if a graph represents a function, we simply need to see if any vertical lines hit the graph in more than one point. If it does, the graph is not a function.

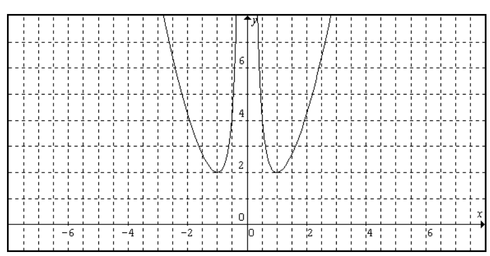
<u>Example 4</u>:

Determine if the graph represents a function. If it is a function, determine the domain and range.

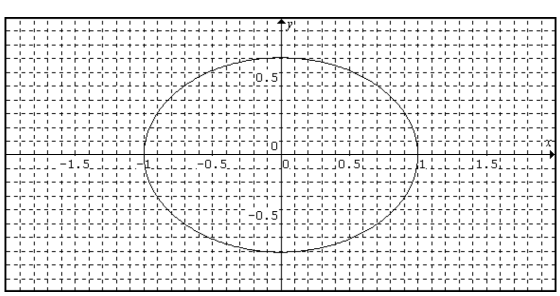
a.



b.



c.



Solution:

Parts a and b both pass the vertical line test since every possible vertical line will hit at most one point. Thus a and b are functions. However, take almost any vertical line through part c. You will get the line hitting two points. Thus, part c fails the vertical line test and is therefore not a function.

Although finding the domain by looking at the graph is helpful, we want to be able to determine the domain of a function without looking at its graph.

This is a relatively simple task. Since the domain is the values that we can put into a function to get a real number out, all we have to do is eliminate all the possible values that would give us a non-real number. That is, get rid of any numbers which give us a negative under the radical (this gives us complex numbers) and which give us a zero in the denominator (this is undefined). Taking care of these two cases will be sufficient for now. However, in the future, some functions we will look at in this course will also have to have a restricted domain. We will see these functions in Chapter 11.

Finding the Domain of a Function
To find the domain of a function, just take care of the following two cases: 1. We cannot have a negative under the radical. 2. We cannot have any zeros in the denominator. The domain is all real numbers except where either of the preceding occur.

Example 5:

Find the domain of the following. Put your answer in interval notation.

a. $f(x)=\dfrac{2x}{x^2-x-6}$ b. $g(x)=\sqrt{x-25}$

Solution:

a. We are only worried about zeros in the denominator or negatives under the radical. In this case, there is no radical, so we only need to know when the denominator is zero.

$f(x)=\dfrac{2x}{x^2-x-6}$ is undefined exactly when $x^2-x-6=0$. So we solve.

$$x^2-x-6=0$$
$$(x-3)(x+2)=0 \qquad \text{(By factoring)}$$
$$x-3=0 \;\; or \;\; x+2=0$$

So $f(x)=\dfrac{2x}{x^2-x-6}$ is undefined when $x=3$ and $x=-2$.

Thus our domain is all real numbers except $x=-2,\ 3$.

In interval notation: $(-\infty,\ -2)\cup(-2,\ 3)\cup(3,\ \infty)$.

b. Again we want to eliminate zeros from the denominator and negative under radicals. But in this case, we have no denominator to worry about. So we want no negatives under the radical. However, that is the same as saying:

The domain of $g(x) = \sqrt{x-25}$ is when $x - 25 \ge 0$. If the portion under the radical is greater or equal to zero, it is certainly not negative.

So we solve the inequality as before.

$$\begin{aligned} x - 25 &\ge 0 \\ x &\ge 25 \end{aligned}$$

Thus, the domain of $g(x) = \sqrt{x-25}$ is $[25, \infty)$.

The last we need to take a look at in our preliminary discussion of graphical topics for functions is we need to talk about even and odd functions as well as two types of symmetry for graphs of functions.

Let's start with the following definition

> **Definition:** A function is called <u>even</u> if $f(-x) = f(x)$ for all values in the domain of f and a function is called <u>odd</u> if $f(-x) = -f(x)$ for all values in the domain of f.

The basic idea is that a function is even if, when we evaluate it at –x it has no change (we get the original function back) or we get the negative of the original function.

Not all functions are either even or odd. Some functions don't satisfy either definition and therefore would be neither even nor odd.

Also, as we will see, whether or not a function is even or odd turns out to give us some very helpful information into the graph of the function. However, let's just get used to working with the definition first.

<u>Example 6</u>:

Determine if the following functions are even, odd, or neither.

a. $f(x) = 6x^2 - 2x^4$ b. $g(x) = \frac{1}{x^3+x}$ c. $h(x) = 2x^3 + x^2 - 1$

Solution:

a. To determine if the function is even, odd, or neither, we need to simply substitute in a –x to the function and see we can make it equal either the original function again, or the negative of the original function.

So we have

$$\begin{aligned} f(-x) &= 6(-x)^2 - 2(-x)^4 \\ &= 6x^2 - 2x^4 \\ &= f(x) \end{aligned}$$

Therefore, since we were able to show $f(-x) = f(x)$, f is an even function.

b. Again, we just need to evaluate the function at –x and see what happens.

This gives us

$$
\begin{aligned}
g(-x) &= \frac{1}{(-x)^3 + (-x)} \\
&= \frac{1}{-x^3 - x} \\
&= \frac{1}{-(x^3 + x)} \\
&= -\frac{1}{x^3 + x} \\
&= -g(x)
\end{aligned}
$$

Since $g(-x) = -g(x)$, we know that $g(x) = \frac{1}{x^3+x}$ is an odd function.

c. Lastly, we again determine if the function is even or odd by plugging in –x.

We get

$$
\begin{aligned}
h(-x) &= 2(-x)^3 + (-x)^2 - 1 \\
&= -2x^3 + x^2 - 1
\end{aligned}
$$

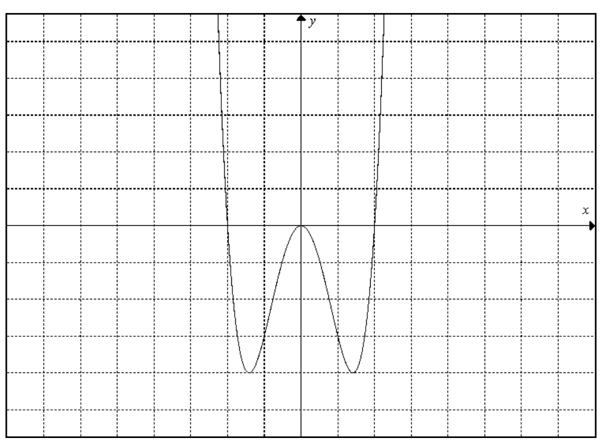
Since there is no way to make this expression equal to either $h(x)$ or $-h(x)$, this function is neither even, nor odd.

Finally, in this section, we want to see how the ideas of even and odd functions affect the graphs of our functions. As it turns out, whether or not a function is even or odd, tells us if and what type of symmetry the graph of our function has.

For a point of reference, a graph is considered symmetric, if the graph is a mirror image across the value of the symmetry (usually a line or a point).

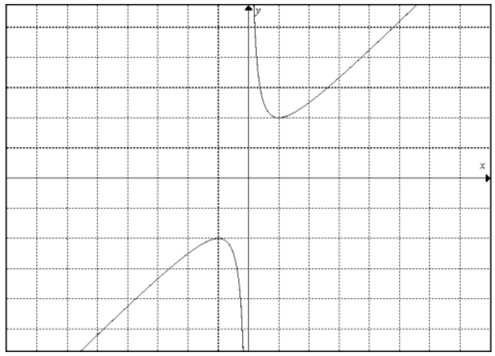
We have numerous types of symmetry that are possible, but for the moment, the only types of symmetry that we are interested in is symmetry about the y-axis, and symmetry about the origin.

If a graph is symmetric about the y-axis, the graph looks like a mirror image of itself, where the mirror is placed on the y-axis. An example of a graph of this type would be



We can see that the graph looks the same on both sides of the y-axis, just mirrored. This is what y-axis symmetry does.

For symmetry about the origin, we need to visualize what would happen if we placed a mirror on just the single point at the origin. We would get something like



Symmetry about the origin is a little harder to visualize, however, we can see that the graph look very much the same on both the top and bottom but has a very clear mirrored quality. This is the idea of symmetry about the origin.

Now that we have an idea about what these two types of symmetry look like, how do we determine if we have either of these symmetries on a particular function.

Symmetry about the y-axis and origin
1. If a function is even, then it is symmetric about the y-axis. 2. If a function is odd, then it is symmetric about the origin.

So, to determine what type of symmetry a graph of a function has, we just need to evaluate the function at –x and see what we get. If $f(-x) = f(x)$ then the function is symmetric about the y-axis, and if $f(-x) = -f(x)$ then the function is symmetric about the origin.

Lets see an example

Example 7:

Determine the type of symmetry of each function.

a. $f(x) = 6x - 2x^3$ b. $g(x) = \frac{1}{x^4}$ c. $h(x) = \frac{1}{x} + x$

Solution:

a. To determine the type of symmetry we have, we just evaluate the function for –x
We get,

$$\begin{aligned} f(-x) &= 6(-x) - 2(-x)^3 \\ &= -6x + 2x^3 \\ &= -(6x - 2x^3) \\ &= -f(x) \end{aligned}$$

Therefore, since the function is odd, it is symmetric about the origin.

b. Again, we will evaluate at –x and see what we get.

$$g(-x) = \frac{1}{(-x)^4}$$
$$= \frac{1}{x^4} = g(x)$$

Since the function is even, this means that it is symmetric about the y-axis.

c. Finally, we again test for symmetry the same way we did the others.

$$h(-x) = \frac{1}{(-x)} + (-x)$$
$$= -\frac{1}{x} - x$$
$$= -\left(\frac{1}{x} + x\right) = -h(x)$$

Since the function is odd, it is symmetric about the origin.

Clearly, since some functions are neither even nor odd, not every function has symmetry. In fact, most functions do not. But if a function has symmetry, we would like to know since it would make graphing it much more precise.

9.3 Exercises

Graph the following functions. Determine the domain and range of each.

1. $f(x) = 3x - 1$
2. $g(x) = 7x + 2$
3. $h(x) = \frac{1}{2}x + 4$
4. $f(x) = \frac{1}{3}x - \frac{1}{4}$
5. $f(x) = \sqrt{2x}$
6. $g(x) = \sqrt{1-x}$
7. $h(x) = \sqrt[3]{x+1}$
8. $F(x) = -\sqrt{x+2}$
9. $f(x) = |x-3|$
10. $h(x) = |2x| + 2$
11. $f(x) = |x| - 2$
12. $g(x) = |x+1| - 1$
13. $f(x) = |x-1| + |x+1|$
14. $f(x) = |x+2| - |x-1|$
15. $g(x) = 2x^2$
16. $g(x) = 1 - x^2$
17. $h(x) = x^3$
18. $p(x) = x^2 + x - 1$
19. $q(x) = x^2 + 2x + 1$
20. $f(x) = x^3 + 2$
21. $f(x) = x^4 - 3x^3$
22. $h(x) = x^3 - 7x + 1$
23. $f(x) = |x^2 - 1|$
24. $g(x) = |x^3|$
25. $r(x) = \dfrac{1}{x}$
26. $S(x) = \dfrac{x}{x+1}$
27. $f(x) = \sqrt{|1-x|}$
28. $g(x) = 1 - (x-4)^2$
29. $h(x) = -\sqrt{-x} + 2$
30. $g(x) = (x-1)(x+1)(x-2)$

31. Graph the following functions.
    a. $f(x) = x^2$ b. $f(x) = (x-2)^2$ c. $f(x) = x^2 - 2$
    d. $f(x) = -x^2$ e. $f(x) = -x^2 - 2$ f. $f(x) = (x-2)^2 - 2$

    What is the relationship between parts a. and b.? a. and c.? a. and d.? a. and e.? a. and f.?

32. Graph the following functions.

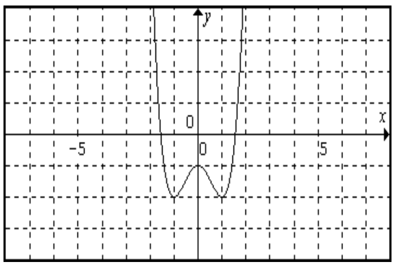
a. $g(x)=|x|$ b. $g(x)=|x+2|$ c. $g(x)=|x|+2$

d. $g(x)=-|x|$ e. $g(x)=-|x+2|$ f. $g(x)=|x+2|+2$

What is the relationship between parts a. and b.? a. and c.? a. and d.? a. and e.? a. and f.?

Determine if the following graph represents a function.

33.



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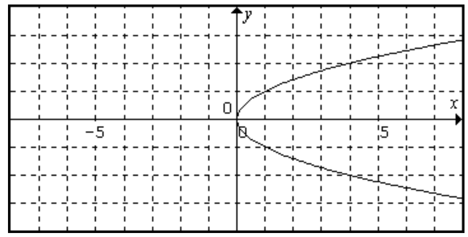

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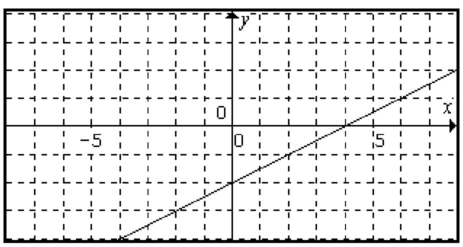

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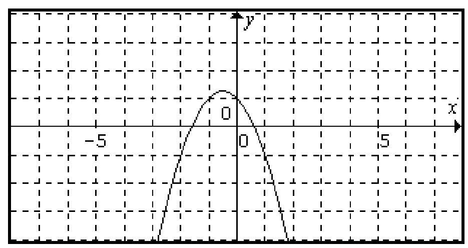

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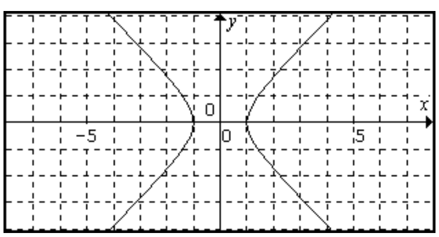

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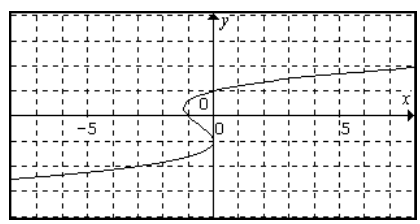

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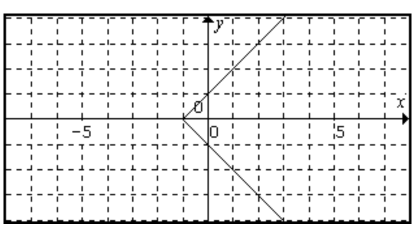

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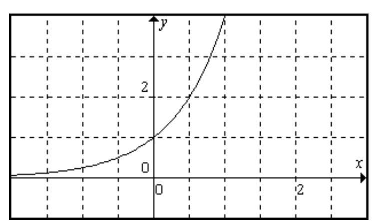

Find the domain in interval notation of the following functions.

41. $h(x)=\sqrt{2x-1}$

42. $f(x)=|x-4|$

43. $f(x)=4x-3$

44. $g(x)=\sqrt{x+2}$

45. $f(x)=\dfrac{9}{x^2+1}$

46. $h(x)=\frac{x}{x^2-1}$

47. $g(t)=\frac{t+3}{t^2+2t}$

48. $f(x)=\frac{x-x^2}{x^2-7x-8}$

49. $h(x)=\sqrt{x+4}$

50. $f(x)=\sqrt{4x-3}$

51. $g(x)=\sqrt{2-x}$

52. $f(t)=\frac{t^{30}}{t^3-t^2}$

53. $f(x)=\sqrt{5-2x}$

54. $g(x)=x^4-7x^3+14$

55. $f(x)=\frac{x+1}{x-4}$

56. $g(x)=1-\sqrt{3-x}$

57. $h(l)=\frac{2l}{l^2-4}$

58. $f(x)=\frac{x}{\sqrt{x}}$

59. $g(x)=\sqrt{-4x}$

60. $g(x)=\frac{\sqrt{x-1}}{2x+1}$

61. $h(x)=\frac{\sqrt{4x+2}}{x-1}$

62. $f(x)=\frac{1}{1-\sqrt{x}}$

63. $f(x)=\frac{x}{\sqrt{x+1}}$

64. $f(x)=\frac{1-\sqrt{x}}{\sqrt{x}}$

Determine if the following functions are even, odd, or neither.

65. $f(x)=x^2$

66. $g(x)=x^3$

67. $h(x)=-2x^3+x$

68. $f(x)=3x^4+x^2$

69. $g(x)=x^2+2x$

70. $h(x)=3x^3+2x^2+x$

71. $f(x)=\frac{1}{x}$

72. $g(x)=\frac{1}{2x^2}$

73. $h(x)=|x|$

74. $f(x)=-4x^3+3x^2$

75. $g(x)=6x^3++x^2+x$

76. $h(x)=|x+1|$

77. $f(x)=\frac{1}{x^3+x}$

78. $g(x)=\frac{1}{x^2+x}$

79. $h(x)=\sqrt{x}$

80. $f(x)=\sqrt{x+1}$

81. $g(x)=x^2+|x|$

82. $h(x)=\frac{1}{x}+x$

Determine the type of symmetry of each function.

83. $f(x)=x^2$

84. $g(x)=x^3$

85. $h(x)=\frac{1}{x}$

86. $f(x)=-2x^4+x^2$

87. $g(x)=|x|$

88. $h(x)=\frac{1}{x^2}$

89. $f(x)=x^3-x$

90. $g(x)=|2x|+1$

91. $h(x)=\sqrt[3]{x}$

92. $f(x)=\sqrt{|x|}$

93. $g(x)=|x-1|+|x+1|$

94. $h(x)=x^4-2x^2$

95. $f(x)=\frac{1}{x^2-1}$

96. $g(x)=\frac{x}{x^4-1}$