9.2 Introduction to Functions

Now that we have reviewed the basic concepts of graphing that we learned earlier in the book we want to turn our attention to one of the most important concepts in algebra, the idea of Functions.

In order to discuss functions we first consider the following definitions:

Definitions:
Relation- any set of ordered pairs
Domain- the set of first coordinates of the ordered pairs in a relation
Range- the set of second coordinates of the ordered pairs in a relation

Example 1

The following is a relation $\{(0,1),(1,0),(2,4),(2,3)\}$.
The domain of this relation is $\{0,1,2\}$, and the range is $\{1,0,3,4\}$. Recall that we do not repeat values in a set. This is the reason that 2 only shows up once in the domain.

Now that we have a handle on relations we would like to talk about a very special type of relation. The function.

Definition: Function- a relation where no two ordered pairs have the same first coordinate.

Our relation in example 1 is clearly not a function since the ordered pairs (2, 4) and (2, 3) have the same first coordinate.

Example 2

Consider the following relation $\{(A,4),(B,3),(C,2),(D,1),(F,0)\}$.
Find the domain and range. Is the relation a function?

Solution:
The domain is the set of first coordinates. So the domain is $\{A,B,C,D,F\}$.
The range is the set of second coordinates. So the range is $\{0,1,2,3,4\}$.
This relation is clearly a function since the first coordinate never repeats.

So, we can see that a function is a relationship between the domain and range. In fact, we can define a function in a much more general way. As a relationship between two sets.

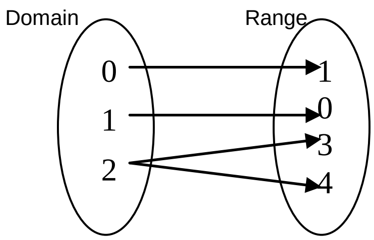
Definition: Function- a rule of correspondence between two sets that assigns each element of the first set to **exactly one** element of the second set. The first set is the domain and the second set contains the range.

Defining a function in this way we can talk about functions in a much broader sense. The function idea is a very important one. What this definition really means is every one object in the domain is assigned only one object of the range.

Many things in today's world follow this idea. For example, your calculator is a function. For every one button you type, you expect one thing to come up on the screen. If it was not a function you could type a 2 and get 3 or 4. Instead we want to type a 2 and get a 2.

One way to view this idea of "one thing in gives you one thing out" is with a visual representation of sets and relations.

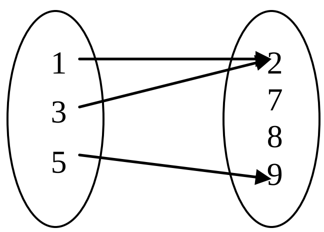
Recall our relation from example 1: $\{(0,1),(1,0),(2,4),(2,3)\}$. If we think of the ordered pairs as assignments, i.e. 0 goes to 1, 1 goes to 0, 2 goes to 3 and 2 goes to 4, we could draw this relation as follows



With this representation of a relation we can see clearly that the 2 gets assigned to two different places. This is a violation of the function idea, since each item in the domain can get assigned to only one item in the range.

<u>Example 3</u>

Is the following a function? Write as a relation of ordered pairs.



Solution:

Even though the value 2 has two assignments at it, the graph is a function. The reason is that by definition, a relation is a function when one domain member is assigned to two range members. The graph above shows one range member coming from two different domain members. Its like saying 1+1=2 and 3-1=2. They both give the same answer but come at it from different directions. It wouldn't be true however that 1+1=2 and 1+1=3. A function requires one thing in gives you only one thing out.

The associated relation is $\{(1,2),(3,2),(5,9)\}$.

Now recall that the last section dealt with linear equations in two variables. Each of these equations produces a collection of ordered pairs that we graph and connect. So this tells us that equations in two variables also represent relations. Therefore, it is natural for us to ask if these equations are functions or not.

<u>Example 4</u>:

Which of the following represent a function?

a. $x+y=1$ b. $y=2x^2-3$ c. $|y|=x+2$

Solution:

Since ordered pairs are listed as (x, y), we assume that the x's represent the domain.

a. The question of whether or not $x + y = 1$ is a function can be simplified to the following: Does each value of x, give exactly one value of y? For this equation it seems rather simple to see that for any value of x, y is simply 1-x. There is no way the expression 1-x can give more than one value for any one value of x. Therefore, $x + y = 1$ is a function.

b. Again the question is, does each value of x in $y = 2x^2 - 3$, give exactly one value of y? Since the only thing being done to x is its squared, multiplied by 2 then minus 3, it seems clear that you can only get one value of y for a specific value of x. Thus, $y = 2x^2 - 3$ is a function.

c. Finally, does each value of x in $|y| = x + 2$, give exactly one value of y? This is more difficult to see. Lets start by taking any value of x, say x=1. We get the following

$$|y| = 1 + 2$$
$$|y| = 3$$

Which is an absolute value equation with solutions $y = 3$ and $y = -3$. So the one value x=1 gives two values of y. Thereby violating the function notion. Thus, $|y| = x + 2$ is not a function.

A good rule of thumb is if stuff is being done to the y variable, then the equation is less likely to be a function.

In the function $y = -x + 1$ from example 4a, we call x the <u>independent</u> variable and y the <u>dependant</u> variable since the y values depend on the x value given.

Since functions are such an important idea, they have a special notation.
We write,

$f(x) = -x + 1$ instead of $y = -x + 1$ and we say "f of x equals –x+1".

f is the <u>name</u> of the function, x is the <u>domain value</u> and $f(x)$ is the <u>range value</u> for a given x.
When we see this notation we can assume that the equation really does represent a function.
We do not need to verify that it really is a function.

Now that we have the proper notation for functions we need to learn to evaluate functions for specific domain values and expressions. In order to do this we use a simple substitution concept.

<u>Example 5</u>:

Evaluate the following for the function $f(x) = 3 - 7x$.

a. $f(-1)$ b. $f(w)$ c. $f(a+2)$ d. $f(x+h)$

Solution:

a. Notice that the difference between $f(x)$ and $f(-1)$ is that the x got replaced with –1. So in the expression $f(x)=3-7x$ we should replace x with –1. We get

$$\begin{aligned} f(x) &= 3-7x \\ f(-1) &= 3-7(-1) \\ &= 3+7 \\ &= 10 \end{aligned}$$

So, $f(-1)=10$.

b. Just like the previous example we notice that the x is replaced by a w. Thus, we have

$$f(w)=3-7w$$

c. This time the x gets replaced by a much larger expression. That is a+2. The procedure, however, is exactly the same. We simply replace the x in $f(x)=3-7x$ with a+2 and simplify.

$$\begin{aligned} f(a+2) &= 3-7(a+2) \\ &= 3-7a-14 \\ &= -11-7a \end{aligned}$$

So, $f(a+2)=-11-7a$.

d. Finally, similar to the previous example, we need to replace the x with an x+h. This gives

$$\begin{aligned} f(x+h) &= 3-7(x+h) \\ &= 3-7x-7h \end{aligned}$$

<u>Example 6</u>:

Evaluate the following for the function $h(x)=x^2-x$.

a. $h(-2)$ b. $h(t)-h(t^2)$ c. $\dfrac{h(x+k)-h(x)}{k}$

Solution:

a. Again here we notice the difference in the expressions $h(x)$ and $h(-2)$ is that the x gets changed to a –2. So in $h(x)=x^2-x$ we need to replace x with –2 and simplify. We get

$$\begin{aligned} h(x) &= x^2-x \\ h(-2) &= (-2)^2-(-2) \\ &= 4+2 \\ &= 6 \end{aligned}$$

So, $h(-2)=6$.

b. This time we have two different expressions to deal with, $h(t)$ and $h(t^2)$. However, the process is the same. We make the individual substitutions and simplify.

$$
\begin{aligned}
h(t)-h(t^2) &= (t^2-t)-((t^2)^2-t^2) \\
&= t^2-t-t^4+t^2 \\
&= -t^4+2t^2-t
\end{aligned}
$$

So, $h(t)-h(t^2)=-t^4+2t^2-t$.

c. Lastly, in the expression $\dfrac{h(x+k)-h(x)}{k}$ we have again several different expressions combined to make a larger expression. We can simply deal with them as they show up. We start with the $h(x+k)$. We need to replace the x with an x+k. This gives us $h(x+k)=(x+k)^2-(x+k)$. Then we have $h(x)$ which is just the function we started with. Combining this over k and simplifying we have

$$
\begin{aligned}
\frac{h(x+k)-h(x)}{k} &= \frac{(x+k)^2-(x+k)-(x^2-x)}{k} \\
&= \frac{x^2+2xk+k^2-x-k-x^2+x}{k} \\
&= \frac{2xk+k^2-k}{k} \\
&= \frac{k(2x+k-1)}{k} \\
&= 2x+k-1
\end{aligned}
$$

The expression $\dfrac{h(x+k)-h(x)}{k}$ in the last example is called the <u>difference quotient</u> which has many very important applications in mathematics. If done correctly, the denominator, k in this case, should cancel out in the end.

<u>Example 7</u>:

Find the following difference quotients:

a. $\dfrac{g(x+h)-g(x)}{h}$ for $g(x)=3x-4$

b. $\dfrac{F(x+h)-F(x)}{h}$ for $F(x)=\dfrac{x}{x+1}$

Solution:

a. Just like the previous example we need to insert all the pieces carefully and then simplify.

$$\frac{g(x+h)-g(x)}{h}=\frac{3(x+h)-4-(3x-4)}{h}$$
$$=\frac{3x+3h-4-3x+4}{h}$$
$$=\frac{3h}{h}$$
$$=3$$

b. This example is much more complicated since it will leave us with a complex fraction to simplify. Just recall, however, we can simplify a complex fraction by multiplying the numerator and denominator by the LCD.

$$\frac{F(x+h)-F(x)}{h}=\frac{\frac{(x+h)}{(x+h)+1}-\frac{x}{x+1}}{h} \qquad LCD=(x+h+1)(x+1)$$
$$=\frac{\left(\frac{x+h}{x+h+1}-\frac{x}{x+1}\right)(x+h+1)(x+1)}{h(x+h+1)(x+1)}$$
$$=\frac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}$$
$$=\frac{x^2+x+xh+h-x^2-xh-x}{h(x+h+1)(x+1)}$$
$$=\frac{h}{h(x+h+1)(x+1)}$$
$$=\frac{1}{(x+h+1)(x+1)}$$

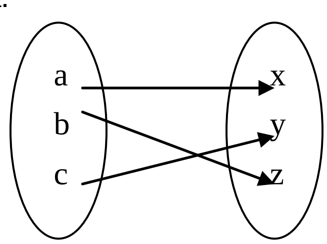
9.2 Exercises

1. Determine which of the following relations is a function. Find the domain and range in each.
   a. $\{(2,1),(0,0),(6,5),(3,2),(4,-1)\}$
   b. $\{(1,1),(1,0),(2,7),(-3,0)\}$
   c. $\{(a,A),(e,E),(i,I),(o,O),(u,U)\}$
   d. $\{(x_1,y_1),(x_2,y_2),(x_3,y_1),(x_4,y_2)\}$

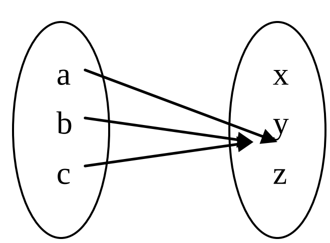
2. Determine which of the following relations is a function. Find the domain and range in each.
   a. $\{(1,1),(1,2),(2,1),(2,2)\}$
   b. $\{(-1,1),(1,1),(3,4),(-3,-4)\}$
   c. $\{(a,\alpha),(b,\beta),(c,\chi)\}$
   d. $\{(x,y),(y,z),(z,y),(x,z)\}$

3. Which of the following are functions?

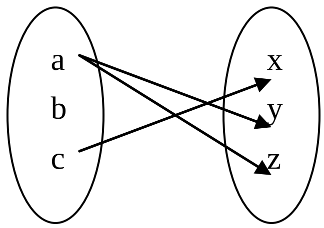
   a.



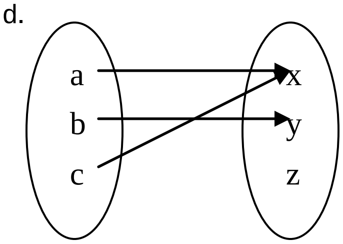
   b.



c.

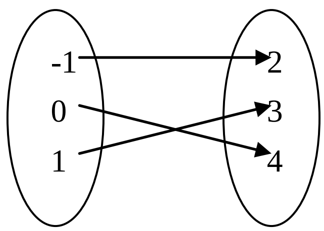


d.

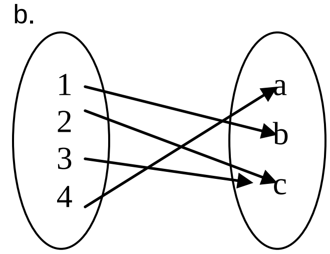


4. Which of the following are functions?

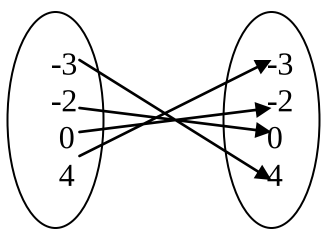
a.



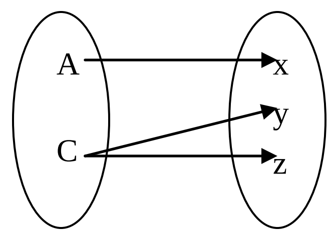
b.



c.



d.



Determine if the following equations represent a function.

5. $3x+y=7$
6. $2y-x=12$
7. $y=x^3-7x$
8. $x=y^2+1$
9. $2y+|x-1|=3$
10. $|y-1|+x=0$
11. $x^2=4y-x$
12. $y=2y-3x$
13. $y=\sqrt{2-3x}$
14. $y=\sqrt{x+1}$

15. Let $f(x)=3x-1$. Evaluate the following.
   a. $f(0)$
   b. $f(1)$
   c. $f(-1)$
   d. $f(-2)$

16. Let $g(x)=2-7x$. Evaluate the following.
   a. $g(0)$
   b. $g(-1)$
   c. $g\left(\frac{1}{7}\right)$
   d. $g(1)$

17. Let $h(x)=2x^2-x$. Evaluate the following.
   a. $h(0)$
   b. $h(-2)$
   c. $h\left(\frac{1}{2}\right)$
   d. $h(1)$

18. Let $f(x)=3-5x^2$. Evaluate the following.
   a. $f(0)$
   b. $f(1)$
   c. $f(-1)$
   d. $f\left(\frac{1}{\sqrt{5}}\right)$

19. Let $g(x)=|2x-3|$. Evaluate the following.
   a. $g(3)$
   b. $g(0)$
   c. $g(-1)$
   d. $g(a)$

20. Let $h(x)=|x-1|+|x+1|$. Evaluate the following.
   a. $h(0)$
   b. $h(1)$
   c. $h(-1)$
   d. $h(y)$

21. Let $f(x)=\dfrac{2x-1}{x+1}$. Evaluate the following.
   a. $f(0)$
   b. $f(1)$
   c. $f(2)$
   d. $f(t)$

22. Let $g(x)=\dfrac{1}{x}+1$. Evaluate the following.
   a. $g(1)$
   b. $g(-1)$
   c. $g(2)$
   d. $g(b)$

23. Let $h(x)=\sqrt{4x+1}$. Evaluate the following.
   a. $h(0)$
   b. $h\left(\frac{3}{4}\right)$
   c. $h(n)$
   d. $h(a+b)$

24. Let $f(x)=\sqrt{2-x}+1$. Evaluate the following.
   a. $f(0)$
   b. $f(-2)$
   c. $f(t)$
   d. $f(t+1)$

25. Let $f(x)=2x-|x|$. Evaluate the following.
   a. $f(1)$
   b. $f(-1)$
   c. $f(t)$
   d. $f(t-2)$

26. Let $f(x)=x^2+x-1$. Evaluate the following.
   a. $f(0)$
   b. $f(-2)$
   c. $f(a)$
   d. $f(b-1)$

27. Let $f(t)=\sqrt{2t-1}$. Evaluate the following.
   a. $f\left(\frac{1}{2}\right)$
   b. $f(apple)$
   c. $f(a+1)$

28. Let $f(a)=|3a-3|$. Evaluate the following.
   a. $f\left(\frac{1}{3}\right)$
   b. $f(whatever)$
   c. $f(a+h)$

29. Let $p(x)=2-3x$. Evaluate the following.
   a. $p(-1)$
   b. $p(u)$
   c. $p(x+1)$
   d. $\dfrac{p(x+h)-p(x)}{h}$

30. Let $g(x)=7x-3$. Evaluate the following.
    a. $g(-1)$
    b. $g(n)$
    c. $g(x+2)$
    d. $\dfrac{g(x+h)-g(x)}{h}$

31. Let $h(x)=3x^2-x$. Evaluate the following.
    a. $h(t)$
    b. $h(t+1)$
    c. $h(t^2+2t)$
    d. $\dfrac{h(x+k)-h(x)}{k}$

32. Let $h(x)=1-x-x^2$. Evaluate the following.
    a. $h(-4)$
    b. $h(c+1)$
    c. $h(\sqrt{t}+2t)$
    d. $\dfrac{h(x+k)-h(x)}{k}$

33. Let $f(x)=2x^2-3x+1$. Evaluate the following.
    a. $f(-x)$
    b. $f(2x-1)$
    c. $f(\sqrt{x+2})$
    d. $\dfrac{f(x+h)-f(x)}{h}$

34. Let $f(x)=3x^2+2x-5$. Evaluate the following.
    a. $f(1-x)$
    b. $f(\sqrt{x}-1)$
    c. $f(x+h)$
    d. $\dfrac{f(x+h)-f(x)}{h}$

35. Let $g(x)=ax^2+bx+c$. Evaluate the following.
    a. $g(\Delta)$
    b. $g(\Delta+1)$
    c. $g(2\Delta+\Theta)$
    d. $\dfrac{g(x+h)-g(x)}{h}$

36. Let $g(x)=\dfrac{1}{x+1}$. Evaluate the following.
    a. $g(2x)$
    b. $g(x-1)$
    c. $g(a-3b)$
    d. $\dfrac{g(x+h)-g(x)}{h}$

37. Let $g(x)=\dfrac{1}{x^2}$. Evaluate the following.
    a. $g\left(\dfrac{1}{x}\right)$
    b. $g(n-m)$
    c. $g(t^2-t)$
    d. $\dfrac{g(x+h)-g(x)}{h}$

38. Let $f(x)=\dfrac{2x}{3x-2}$. Evaluate the following.
    a. $f\left(\dfrac{x}{y}\right)$
    b. $f(x-y)$
    c. $f\left(\dfrac{2x}{3x-2}\right)$
    d. $\dfrac{f(x+h)-f(x)}{h}$

39. Let $f(x)=\dfrac{x}{x+1}$. Evaluate the following.
   a. $f(x^2)$
   b. $f(n-1)$
   c. $f\left(\dfrac{x}{1-x}\right)$
   d. $\dfrac{f(x+h)-f(x)}{h}$

40. Let $h(x)=\dfrac{1}{x^2-1}$. Evaluate the following.
   a. $h(\sqrt{x})$
   b. $h(x+k)$
   c. $\dfrac{h(x+k)-h(x)}{k}$

41. Let $h(x)=x^3$. Evaluate the following.
   a. $h(\sqrt[3]{x})$
   b. $h(x+k)$
   c. $\dfrac{h(x+k)-h(x)}{k}$

42. Let $g(x)=x^4$. Evaluate the following.
   a. $g(-1)$
   b. $g(x+h)$
   c. $\dfrac{g(x+h)-g(x)}{h}$

43. Let $g(x)=x^3-2x^2-2x+1$. Evaluate the following.
   a. $g(-1)$
   b. $g(x+h)$
   c. $\dfrac{g(x+h)-g(x)}{h}$