

8.7 Solving Radical Equations

In this section we want to learn how to solve equations containing radicals, like $\sqrt{5x-4} = 9$. In order to do this we need the following property.

n-th Power Property

If $a = b$, then $a^n = b^n$.

Basically, this property tells us we can raise both sides of any equation to any power we would like. However, we must be careful. There are several places where we can make serious mistakes.

First, we need to make sure that we really are raising the entire side to the n th power and not just each term individually. Also, when using the n th power property there is the possibility that we end up with solutions that don't check, called extraneous solutions. The reason we get these on occasion has to do with the logical construction of the property. The property tells us that if something is a solution to $a = b$ then it must also be a solution to $a^n = b^n$. That doesn't mean that if something is a solution to $a^n = b^n$ that it is a solution to $a = b$. To rectify this, we simply check our answers and throw away the ones that do not work.

So, always remember, be careful to raise the entire side to the n -th power, and **we must always check our answers**.

Example 1:

Solve.

a. $\sqrt{5x-4} = 9$

b. $\sqrt[3]{3-2x} = -2$

c. $\sqrt[4]{3-x} = -1$

Solution:

- a. We can use the n -th power property to get rid of the radical. Since we have a square root, we should use the second power, i.e. we should square both sides of the equation. Then we can solve like usual. We get

$$\sqrt{5x-4} = 9$$

$$(\sqrt{5x-4})^2 = (9)^2$$

$$5x-4 = 81$$

$$5x = 85$$

$$x = 17$$

Now we must check our answer. We do this in the original equation. If the value does not check, then we eliminate it and would have no solution. We get

$$\sqrt{5(17)-4} = 9$$

$$\sqrt{85-4} = 9$$

$$\sqrt{81} = 9$$

$$9 = 9$$

Since the value checks, our solution is set $\{17\}$.

- b. Again, we need to raise both sides of the equation to a suitable power to get rid of the radical. This time we will use the 3rd power since we have a cube root. We get

$$\begin{aligned}\sqrt[3]{3-2x} &= -2 \\ (\sqrt[3]{3-2x})^3 &= (-2)^3 \\ 3-2x &= -8 \\ -2x &= -11 \\ x &= \frac{11}{2}\end{aligned}$$

Now we check the solution.

$$\begin{aligned}\sqrt[3]{3-2\left(\frac{11}{2}\right)} &= -2 \\ \sqrt[3]{3-11} &= -2 \\ \sqrt[3]{-8} &= -2 \\ -2 &= -2\end{aligned}$$

Since the value checks the solution set is $\left\{\frac{11}{2}\right\}$.

- c. Finally, we need to raise both sides to the 4th power since we have a 4th root. We get

$$\begin{aligned}\sqrt[4]{3-x} &= -1 \\ (\sqrt[4]{3-x})^4 &= (-1)^4 \\ 3-x &= 1 \\ -x &= -2 \\ x &= 2\end{aligned}$$

Check:

$$\begin{aligned}\sqrt[4]{3-2} &= -1 \\ \sqrt[4]{1} &= -1 \\ 1 &\neq -1\end{aligned}$$

Since it does not check, 2 can not be a solution to the equation. Therefore, we have the equation has no solution.

Now, sometimes the equation is a bit more complicated. What we need to know is that before raising both sides to the n-th power, we must always isolate the radical expression first. That way we are assured that it will cancel when we use the exponent. Also, sometimes, after raising both sides to the n-th power, we end up with a radical still in the equation. In this situation, we need to again isolate the radical and again use the exponent. We keep repeating this process until all the radicals are gone. Then we solve as usual.

Example 2:

Solve.

a. $\sqrt{2x-3}-2=1$

b. $\sqrt{x+1}=2-\sqrt{x}$

c. $\sqrt{2x+5}-\sqrt{3x-2}=1$

Solution:

- a. First thing we need to do is isolate the radical. Once we have done that we can square both sides of the equation as we did in example1. Then, of course, we finish solving and check.

$$\begin{aligned}
\sqrt{2x-3}-2 &= 1 \\
\sqrt{2x-3} &= 3 \\
(\sqrt{2x-3})^2 &= (3)^2 \\
2x-3 &= 9 \\
2x &= 12 \\
x &= 6
\end{aligned}$$

Check:

$$\begin{aligned}
\sqrt{2(6)-3}-2 &= 1 \\
\sqrt{12-3}-2 &= 1 \\
\sqrt{9}-2 &= 1 \\
3-2 &= 1 \\
1 &= 1
\end{aligned}$$

The value checks. Therefore the solution set is $\{6\}$.

- b. Notice this time that one of the radicals is already isolated. Therefore we can go ahead and square both sides. Must, however be very careful with squaring the right side since it has two terms. It generally makes it easier to square properly by writing the entire side out twice and then multiplying as we learned in section 8.4. We get

$$\begin{aligned}
\sqrt{x+1} &= 2-\sqrt{x} \\
(\sqrt{x+1})^2 &= (2-\sqrt{x})^2 \\
x+1 &= (2-\sqrt{x})(2-\sqrt{x}) \\
x+1 &= 4-2\sqrt{x}-2\sqrt{x}+(\sqrt{x})^2 \\
x+1 &= 4-4\sqrt{x}+x
\end{aligned}$$

Notice that we are left with another equation that has a radical. Therefore, we need to isolate again and square again. In this case however, it is easier to simply isolate the term containing the radical and the carefully square both sides. We can then continue solving as usual.

$$\begin{aligned}
x+1 &= 4-4\sqrt{x}+x \\
-x-4 &\quad -4 \quad \quad -x \\
-3 &= -4\sqrt{x} \\
(-3)^2 &= (-4\sqrt{x})^2 \\
9 &= 16x \\
\frac{9}{16} &= x
\end{aligned}$$

Check:

$$\begin{aligned}
\sqrt{\frac{9}{16}+1} &= 2-\sqrt{\frac{9}{16}} \\
\sqrt{\frac{25}{16}} &= 2-\frac{3}{4} \\
\frac{5}{4} &= \frac{5}{4}
\end{aligned}$$

Since the solution checks, the solution set is $\left\{\frac{9}{16}\right\}$.

- c. Finally, we must start by isolating one of the two radicals in this equation. We will choose to isolate the positive one. Generally we isolate the more complicated radical in order to eliminate it first. However, either could be isolated and still produce the correct solutions. Once we have isolated the radical then we can square both sides very carefully and then continue as usual. We get

$$\begin{aligned}\sqrt{2x+5} - \sqrt{3x-2} &= 1 \\ (\sqrt{2x+5})^2 &= (\sqrt{3x-2} + 1)^2 \\ 2x+5 &= (\sqrt{3x-2} + 1)(\sqrt{3x-2} + 1) \\ 2x+5 &= (\sqrt{3x-2})^2 + \sqrt{3x-2} + \sqrt{3x-2} + 1 \\ 2x+5 &= 3x-2 + 2\sqrt{3x-2} + 1 \\ 2x+5 &= 3x-1 + 2\sqrt{3x-2} \\ -x+6 &= 2\sqrt{3x-2} \\ (-x+6)^2 &= (2\sqrt{3x-2})^2 \\ (-x+6)(-x+6) &= 4(3x-2) \\ x^2 - 6x - 6x + 36 &= 12x - 8 \\ x^2 - 12x + 36 &= 12x - 8\end{aligned}$$

Recall, to solve an equation that has a squared term we must solve by factoring. That is, we get all terms on one side, factor and set each factor to zero. This gives

$$\begin{aligned}x^2 - 12x + 36 &= 12x - 8 \\ x^2 - 24x + 44 &= 0 \\ (x-22)(x-2) &= 0 \\ x-22 = 0 \quad \text{and} \quad x-2 &= 0 \\ x = 22 \quad \quad \quad x &= 2\end{aligned}$$

Now we must check both answers and keep only those which work

Check:

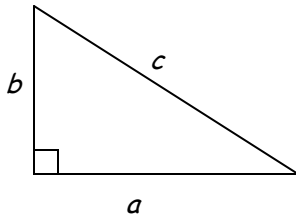
$$\begin{array}{ll}\sqrt{2(22)+5} - \sqrt{3(22)-2} = 1 & \sqrt{2(2)+5} - \sqrt{3(2)-2} = 1 \\ \sqrt{44+5} - \sqrt{66-2} = 1 & \sqrt{4+5} - \sqrt{6-2} = 1 \\ \sqrt{49} - \sqrt{64} = 1 & \sqrt{9} - \sqrt{4} = 1 \\ 7-8 = 1 & 3-2 = 1 \\ -1 \neq 1 & 1 = 1\end{array}$$

So, 22 does not check but 2 does. Therefore, the 22 is an extraneous solution. So our solution set is $\{2\}$.

Finally we want to see some applications of radicals. The first of these requires the Pythagorean theorem. We state it below.

The Pythagorean Theorem

In any right triangle, if a and b are the lengths of the legs of the triangle and c is the length of the hypotenuse, then we have the following



$$a^2 + b^2 = c^2$$

or alternately

$$c = \sqrt{a^2 + b^2}$$

$$a = \sqrt{c^2 - b^2}$$

$$b = \sqrt{c^2 - a^2}$$

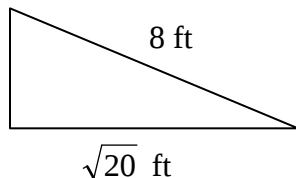
Other types of applications require us to interpret and use a given formula.

Example 3:

Find the missing side of the right triangle if $c = 8 \text{ ft}$, $a = \sqrt{20} \text{ ft}$.

Solution:

First we start by drawing the picture



So we can see that we are clearly missing side b . Therefore we can use the formula

$b = \sqrt{c^2 - a^2}$. We get

$$\begin{aligned} b &= \sqrt{8^2 - (\sqrt{20})^2} \\ &= \sqrt{64 - 20} \\ &= \sqrt{44} \\ &= 2\sqrt{11} \end{aligned}$$

So $b = 2\sqrt{11} \text{ ft} \approx 6.6 \text{ ft}$.

Example 4:

The equation for the time of one pendulum swing (called the period of the pendulum) is given by

$T = 2\pi\sqrt{\frac{L}{32}}$ where T is the time in seconds and L is the length in feet. Find the length of a pendulum of a clock that has a period of 2.3 seconds.

Solution:

We simply need to put the value of 2.3 into the formula for T and solve for L by using the techniques we learned in this section. We proceed as follows

$$T = 2\pi\sqrt{\frac{L}{32}}$$

$$2.3 = 2\pi\sqrt{\frac{L}{32}}$$

$$\frac{2.3}{2\pi} = \sqrt{\frac{L}{32}}$$

$$\left(\frac{2.3}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{32}}\right)^2$$

$$\left(\frac{2.3}{2\pi}\right)^2 = \frac{L}{32}$$

$$L = 32 \cdot \left(\frac{2.3}{2\pi}\right)^2$$

Now we put this into our calculator to get $L \approx 17.15$ feet. Upon checking this value, we find that it does check and therefore the length of the pendulum is 17.15 feet.

8.7 Exercises

Solve.

1. $\sqrt{3x} = 12$
2. $\sqrt{4x} = 2$
3. $\sqrt[3]{1-2x} = -3$
4. $\sqrt[3]{4x-1} = 2$
5. $\sqrt[4]{4x+1} = 2$
6. $\sqrt[4]{2x-9} = 3$
7. $\sqrt{x^2-8x} = 3$
8. $\sqrt{x^2+7x+11} = 1$
9. $\sqrt{8x+1} = x+2$
10. $\sqrt{2(x+5)} = x+5$
11. $\sqrt{3x-5}-2=3$
12. $\sqrt{2x-1}-8=-5$
13. $\sqrt[3]{4x-3}-2=3$
14. $\sqrt[3]{1-3x}+5=3$
15. $1-\sqrt{4x+3}=-5$
16. $7-\sqrt{3x+1}=-1$
17. $\sqrt[3]{x^2+4}-2=0$
18. $\sqrt[3]{x^2+2}-3=0$
19. $\sqrt[4]{x^2+2x+8}-2=0$
20. $\sqrt[4]{x^2+x-1}-1=0$
21. $x=\sqrt{x+7}+5$
22. $3+\sqrt{5-x}=x$
23. $\sqrt{2x-7}=\sqrt{3x-12}$
24. $\sqrt{5x+4}=\sqrt{3x-1}$
25. $4\sqrt{x+1}-x=1$
26. $3\sqrt{x-2}+2=x$
27. $\sqrt{x}+2=x$
28. $x+3\sqrt{x-2}=12$
29. $\sqrt{2+9b}-1=3\sqrt{b}$
30. $\sqrt{x+1}=2-\sqrt{x}$
31. $3+\sqrt{z-6}=\sqrt{z+9}$
32. $\sqrt{4x-3}=2+\sqrt{2x-5}$
33. $\sqrt{5t}=1+\sqrt{5(t-1)}$
34. $\sqrt{2x+4}=3-\sqrt{2x}$
35. $\sqrt{1+6x}+\sqrt{6x}=2$
36. $\sqrt{x}+\sqrt{x-5}=5$
37. $\sqrt{x-17}+\sqrt{x}=17$
38. $\sqrt{x}+\sqrt{x-9}=1$
39. $\sqrt{3x-2}-2\sqrt{x}=-1$
40. $\sqrt{x-1}+\sqrt{x+4}=5$
41. $\sqrt{4x-3}-2=\sqrt{2x-5}$
42. $\sqrt{2x+7}-\sqrt{x+15}=-1$
43. $\sqrt{2x-1}+\sqrt{x}=2$
44. $\sqrt{2x-3}=\sqrt{x+7}-2$
45. $\sqrt{x+2}+\sqrt{3x+4}=2$
46. $\sqrt{6x+7}-\sqrt{3x+3}=1$
47. $\sqrt{x-2}-\sqrt{x+2}+2=0$

$$48. \sqrt{2x-5} - 3\sqrt{x+1} + 7 = 0$$

$$49. \sqrt{x+2} + \sqrt{3x+4} - 2 = 0$$

$$50. 2\sqrt{3x+6} - \sqrt{4x+9} - 5 = 0$$

Solve for the indicated variable.

$$51. v = \sqrt{64d} \text{ for } d$$

$$52. d = \sqrt{1.5h} \text{ for } h$$

$$53. v = \sqrt{2as} \text{ for } a$$

$$54. v = \sqrt{2as} \text{ for } s$$

$$55. t = \sqrt{\frac{2d}{g}} \text{ for } g$$

$$56. t = \sqrt{\frac{2d}{g}} \text{ for } d$$

$$57. y = \sqrt{x-h} + k \text{ for } x$$

$$58. y = \sqrt{x-h} + k \text{ for } h$$

Find the missing side of the right triangle.

$$59. a = 3, b = 5$$

$$60. b = 7, c = 13$$

$$61. b = \sqrt{30}m, c = 20m$$

$$62. a = 10 \text{ yds}, c = \sqrt{120} \text{ yds}$$

63. A ten foot ladder is leaning against a wall. How far is the bottom of the ladder from the wall when the ladder reaches a height of 8 feet?

64. A 25 foot ladder is leaning against a building. How high up the building does the ladder reach if the bottom of the ladder is 5 feet from the building?

65. The diagonal of a TV is 15 inches. The width of the TV is 10 inches. What is the height of the TV?

66. A computer monitor has a width of 13 inches and a height of 10 inches. What is the length of the diagonal of the monitor?

67. A stereo receiver is in a corner of a 12 foot by 14 foot room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If a 4 foot slack is needed at each end, how long of a piece of wire should be used?

68. A baseball diamond is a square which is 90 feet on the side. What is the distance from first base to third base?

69. The formula for the speed of a falling object is $v = \sqrt{64d}$ where v is the speed of the object in feet per second and d is the distance the object has fallen, in feet. What distance has an object fallen if its speed is 150 feet per second?

70. The formula for the speed of a falling object is $v = \sqrt{64d}$ where v is the speed of the object in feet per second and d is the distance the object has fallen, in feet. What distance has an object fallen if its speed is 75 feet per second?

71. The formula for the distance a lookout can see is $d = \sqrt{1.5h}$ where d is the distance in miles and h is the height of the lookout above the water, in feet. How high will the periscope have to go to see a ship that is 6.5 miles away?

72. The formula for the distance a lookout can see is $d = \sqrt{1.5h}$ where d is the distance in miles and h is the height of the lookout above the water, in feet. How high will the periscope have to go to see an island that is 4.3 miles away?

73. The equation for the period of the pendulum is given by $T = 2\pi\sqrt{\frac{L}{32}}$ where T is the time in seconds and L is the length in feet. Find the length of a pendulum of a clock that has a period of 4 seconds.
74. The equation for the period of the pendulum is given by $T = 2\pi\sqrt{\frac{L}{32}}$ where T is the time in seconds and L is the length in feet. Find the length of a pendulum that has a period of 7.2 seconds.
75. The formula for the distance required of a moving object to reach a specific velocity is $v = \sqrt{2as}$ where v is the velocity of the object, a is the acceleration of the object and s is the distance required. What distance is required for an car to reach a velocity of 50 miles per hour if the acceleration is 5 miles per square second?
76. The formula for the distance required of a moving object to reach a specific velocity is $v = \sqrt{2as}$ where v is the velocity of the object, a is the acceleration of the object and s is the distance required. What distance is required for an car to reach a velocity of 72 kilometers per hour if the acceleration is 10 kilometers per square second?
77. The formula for the demand for a certain product is given by $p = 40 - \sqrt{0.01x + 1}$ where x is the number of units demanded per day and p is the price per unit. What is the demand if the price is \$37.55?
78. The formula for the demand for a certain product is given by $p = 40 - \sqrt{0.001x + 1}$ where x is the number of units demanded per day and p is the price per unit. What is the demand if the price is \$34.70?
79. The formula for the escape velocity of a satellite is $v = \sqrt{2gr}\sqrt{\frac{h}{r+h}}$ where v is the velocity, g is the force of gravity, r is the planets radius and h is the height of the satellite above the planet. What is the height above the Earth would the Space Shuttle be if the gravitational pull is 0.0098 km/sec^2 and the escape velocity is 2.45 km/sec ? (The radius of the earth is about 6440 km)
80. The formula for the escape velocity of a satellite is $v = \sqrt{2gr}\sqrt{\frac{h}{r+h}}$ where v is the velocity, g is the force of gravity, r is the planets radius and h is the height of the satellite above the planet. What is the height above the Earth would a satellite be if the gravitational pull is 0.0098 km/sec^2 and the escape velocity is 1.25 km/sec ? (The radius of the earth is about 6440 km)