8.6 The Complex Number System

Earlier in the chapter, we mentioned that we cannot have a negative under a square root, since the square of any positive or negative number is always positive. In this section we want to find a way to deal with an expression that does have a negative under a square root.

First, consider the equation $x^2+1=0$. Clearly this equation has no real number solutions. Therefore, we need to make an entirely new set of numbers to represent these types of values. We start with the following definition

Definition: The imaginary unit
The imaginary unit, i, is defined as $i=\sqrt{-1}$. Therefore, $i^2=-1$.

Also, we can notice that $i^3=i^2\cdot i=-1\cdot i$, $i^4=i^2\cdot i^2=-1\cdot -1=1$ and $i^5=i^4\cdot i=1\cdot i=i$.

In fact, the powers of i continue to repeat like this

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ i^6 &= -1 \\ &\vdots \end{aligned}$$

So we can easily simplify any power of i since every 4th power is just 1. Also, we can see that we should always end up with an expression that has a power of i that is at most 1.

<u>Example 1</u>:

Simplify.

a. i^{12} b. i^{23} c. i^{74}

Solution:

a. By what we saw above we can simply write i^{12} as $\left(i^4\right)^3$. Since $i^4=1$ we have $\left(i^4\right)^3=1^3=1$. Therefore, $i^{12}=1$.

b. Similar to part a. we want to write i^{23} as having as many 4th powers as possible, that way we can have 1 for those values. So can proceed as follows

$$\begin{aligned} i^{23} &= i^{20}\cdot i^3 \\ &= \left(i^4\right)^5\cdot i^3 \\ &= 1^5\cdot -i \qquad \textit{because } i^3=-i \\ &= -i \end{aligned}$$

c. So we will proceed just as we did in part b.

$$
\begin{aligned}
i^{74} &= i^{72} \cdot i^2 \\
&= \left(i^4\right)^{18} \cdot i^2 \\
&= 1^{18} \cdot -1 \qquad \textit{because } i^2 = -1 \\
&= -1
\end{aligned}
$$

Since we are now dealing with a negative under a square root we need to know how to properly simplify radicals with this in mind. So we have the following property.

Property of negative square roots
$\sqrt{-c} = \sqrt{-1 \cdot c} = \sqrt{-1}\sqrt{c} = i\sqrt{c}$

We use this property to simplify square roots that contain negatives.

<u>Example 2</u>:

Simplify.

a. $\sqrt{-9}$ b. $\sqrt{-75}$ c. $\sqrt{25} - \sqrt{-147}$

Solution:

a. We can simply use the property to simplify as follows

$$
\begin{aligned}
\sqrt{-9} &= i\sqrt{9} \\
&= i \cdot 3 \\
&= 3i
\end{aligned}
$$

b. Again, we can "remove" the negative from under the radical by pulling it out as an i and then simplify the resulting expression. We get

$$
\begin{aligned}
\sqrt{-75} &= i\sqrt{75} \\
&= i\sqrt{25 \cdot 3} \\
&= i \cdot 5\sqrt{3} \\
&= 5i\sqrt{3}
\end{aligned}
$$

Note: Textbooks differ on the position of the i in a problem of this type. Most put the i in the back of the expression. However, we choose to put it between the coefficient and the radical to remove any possible confusion of whether or not the i is under the radical. If we were to write as $5\sqrt{3}i$ it can appear that the i is part of the radicand.

c. Again, we will pull or the i and then continue to simplify.

$$
\begin{aligned}
\sqrt{25} - \sqrt{-147} &= \sqrt{25} - i\sqrt{147} \\
&= \sqrt{25} - i\sqrt{49 \cdot 3} \\
&= 5 - 7i\sqrt{3}
\end{aligned}
$$

Since we do not have like radicals we cannot combine the remaining terms and thus are finished.

Now that we have a familiarity with the imaginary unit, we can introduce the number system which it generates.

Definition: Complex Numbers
A number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$ is called a complex number. a is called the real part and b is called the imaginary part. A complex number written with the real part is first and the imaginary part is last is in standard form.

We want to be able to perform basic operations on these complex numbers. Its actually very simple. We simply need to remember that i is really a radical and $i^2=-1$. With this in mind, we can simply add, subtract and multiply as we did in the earlier part of the chapter. We simply need to make sure that we simplify all of our powers of i.

Example 3:

Perform the operations. Put your answers in standard form.

a. $(-10+2i)+(4-7i)$ b. $(-21-50i)-(2+10i)$ c. $10i(8-6i)$

d. $(3+5i)(2-15i)$ e. $\sqrt{-5}\cdot\sqrt{-10}\cdot\sqrt{25}$ f. $\left(5-\sqrt{-12}\right)-\left(9+\sqrt{-108}\right)$

Solution:

a. As we said, we can simply perform operations as we did earlier in this chapter. So that means we need to combine like radicals. In this case, the terms containing i would be like. So we get

$$\begin{aligned}(-10+2i)+(4-7i)&=-10+2i+4-7i\\&=-6-5i\end{aligned}$$

b. This time we need to start by distributing the negative, then combine the like radicals. This gives

$$\begin{aligned}(-21-50i)-(2+10i)&=-21-50i-2-10i\\&=-23-60i\end{aligned}$$

c. Now, to multiply complex numbers it is actually easier to just treat them as polynomials and then just simplify the powers of i by remembering that $i^2=-1$. So we get

$$\begin{aligned}10i(8-6i)&=80i-60i^2\\&=80i-60(-1)\\&=80i+60\\&=60+80i\end{aligned}$$

Since we wanted the answers in standard form, we needed to write the real part first followed by the imaginary part.

d. Again, we will multiply as if these were polynomials and simplify the result. Always remember $i^2=-1$. This gives

$$\begin{aligned}(3+5i)(2-15i)&=6-45i+10i-75i^2\\&=6-35i-75(-1)\\&=6-35i+75\\&=81-35i\end{aligned}$$

e. In an expression of this form, we must **always** start by removing the negatives from the radicals. If we do not we end up with a completely different (and therefore incorrect) result. Once we have the negatives out of the radicals, we can simply multiply as before. We get

$$\begin{aligned}\sqrt{-5}\cdot\sqrt{-10}\cdot\sqrt{25} &= i\sqrt{5}\cdot i\sqrt{10}\cdot\sqrt{25}\\ &= 5i^2\sqrt{50}\\ &= 5(-1)\sqrt{25\cdot 2}\\ &= -5\cdot 5\sqrt{2}\\ &= -25\sqrt{2}\end{aligned}$$

f. Lastly, again we begin by pulling the negatives out of the radicals. Then we simplify as before. This gives

$$\begin{aligned}\left(5-\sqrt{-12}\right)-\left(9+\sqrt{-108}\right) &= \left(5-i\sqrt{12}\right)-\left(9+i\sqrt{108}\right)\\ &= \left(5-2i\sqrt{3}\right)-\left(9+6i\sqrt{3}\right)\\ &= 5-2i\sqrt{3}-9-6i\sqrt{3}\\ &= -4-8i\sqrt{3}\end{aligned}$$

Lastly we need to deal with how to divide complex numbers. However, if we simply remember that i is a radical then we can treat the division as we did before. That is, we just use the conjugate. However this time we have what is called the complex conjugate and complex conjugates always have a very simple product.

Complex Conjugates
$a+bi$ and $a-bi$ are called <u>complex conjugates</u>. Also, $(a+bi)(a-bi)=a^2+b^2$.

So rather that multiplying the conjugates out every time we can simply add the squares of the real and imaginary parts to simplify the process.

<u>Example 4</u>:

Perform the operations. Write your answers in standard form.

a. $\dfrac{17i}{5+3i}$ b. $\dfrac{4-5i}{4+5i}$ c. $\dfrac{1+i}{3i}$ d. $\dfrac{\sqrt{-2}}{\sqrt{12}-\sqrt{-8}}$

Solution:

a. So we can simplify by multiplying numerator and denominator by the conjugate of the denominator as we did before. Then we just need to simplify and reduce. Notice that we can just use the formula above for the product of the conjugates. We proceed as follows

$$\begin{aligned}\frac{17i}{(5+3i)}\frac{(5-3i)}{(5-3i)} &= \frac{85i-51i^2}{5^2+3^2}\\ &= \frac{85i-51(-1)}{25+9}\end{aligned}$$

$$= \frac{51 + 85i}{34}$$

$$= \frac{51}{34} + \frac{85}{34}i$$

$$= \frac{3}{2} + \frac{5}{2}i$$

b. Again, we simply multiply numerator and denominator by the complex conjugate of the denominator and then simplify and reduce. Notice that the conjugate is in fact the numerator, but we need not be concerned with that. We simply multiply it out as we learned before. We get

$$\frac{(4-5i)}{(4+5i)}\frac{(4-5i)}{(4-5i)} = \frac{16 - 20i - 20i + 25i^2}{4^2 + 5^2}$$

$$= \frac{16 - 40i + 25(-1)}{16 + 25}$$

$$= \frac{-9 - 40i}{41}$$

$$= -\frac{9}{41} - \frac{40}{41}i$$

c. This time we only have one term on the denominator. Therefore, there is no need to use the conjugate. We simply need to multiply by something that will eliminate the i. Well, recall $i^2 = -1$. Therefore, we can simply multiply by i on numerator and denominator to eliminate the i on the denominator. We get

$$\frac{(1+i)}{(3i)}\frac{i}{i} = \frac{i + i^2}{3i^2}$$

$$= \frac{i + (-1)}{3(-1)}$$

$$= \frac{-1 + i}{-3}$$

$$= \frac{1}{3} - \frac{1}{3}i$$

d. Lastly, we need to start by pulling out all the negatives and simplifying the radicals. Once that is completed, we can multiply by the conjugate on the numerator and denominator. We get

$$\frac{\sqrt{-2}}{\sqrt{12} - \sqrt{-8}} = \frac{i\sqrt{2}}{\sqrt{12} - i\sqrt{8}}$$

$$= \frac{i\sqrt{2}}{(2\sqrt{3} - 2i\sqrt{2})}\frac{(2\sqrt{3} + 2i\sqrt{2})}{(2\sqrt{3} + 2i\sqrt{2})}$$

$$= \frac{2i\sqrt{6} + 2i^2\sqrt{4}}{\left(2\sqrt{3}\right)^2 + \left(2\sqrt{2}\right)^2}$$

$$= \frac{2i\sqrt{6} + 2(-1)2}{12 + 8}$$

$$= \frac{-4 + 2i\sqrt{6}}{20}$$

$$= \frac{-4}{20} + \frac{2i\sqrt{6}}{20}$$

$$= -\frac{1}{5} + \frac{i\sqrt{6}}{10}$$

8.6 Exercises

Simplify.

1. i^7
2. i^9
3. i^{15}
4. i^{18}
5. i^{29}
6. i^{30}
7. i^{44}
8. i^{56}
9. i^{137}
10. i^{118}
11. $\sqrt{-4}$
12. $\sqrt{-16}$
13. $\sqrt{-25}$
14. $\sqrt{-64}$
15. $\sqrt{-8}$
16. $\sqrt{-27}$
17. $\sqrt{-60}$
18. $\sqrt{-80}$
19. $\sqrt{4} + \sqrt{-4}$
20. $\sqrt{9} + \sqrt{-81}$

Perform the operations. Write your answers in standard form.

21. $(6 + 2i) + (5 - 3i)$
22. $(-7 + 4i) + (4 - i)$
23. $(-5 + 2i) - (6 - 5i)$
24. $(2 - 5i) - (3 - 4i)$
25. $(3 - 6i) - (5 + 4i)$
26. $(-2 + 7i) - (-1 + i)$
27. $(17 + 5i) + (18 - 5i)$
28. $(20 - 4i) + (30 + 4i)$
29. $(15 - 3i) - (14 - 3i)$
30. $(12 - 5i) - (13 - 5i)$
31. $\left(7 + \sqrt{-25}\right) + \left(2 - \sqrt{-9}\right)$
32. $\left(2 - \sqrt{-16}\right) + \left(1 - \sqrt{-81}\right)$
33. $\left(1 - \sqrt{-2}\right) - \left(5 - \sqrt{-18}\right)$
34. $\left(\sqrt{2} - \sqrt{-3}\right) - \left(\sqrt{8} - \sqrt{-12}\right)$
35. $(5i)(2i)$
36. $(6i)(i)$
37. $2(10 - 6i)$
38. $3(5 - 2i)$
39. $2i(4 - i)$
40. $3i(7 - 2i)$
41. $i\sqrt{2}\left(\sqrt{2} - i\sqrt{3}\right)$
42. $i\sqrt{7}\left(2\sqrt{7} - 3i\sqrt{7}\right)$
43. $\sqrt{-3}\left(\sqrt{-3} + \sqrt{-4}\right)$
44. $\sqrt{-12}\left(\sqrt{-3} + \sqrt{-12}\right)$
45. $(-7 + 4i)(4 - i)$
46. $(-5 + 2i)(6 - 5i)$
47. $(5 - 2i)(4 + 3i)$
48. $(6 - 3i)(4 + 5i)$
49. $i(2 + 3i)^2$
50. $2i(3 - i)^2$
51. $\left(7 + \sqrt{-25}\right)\left(2 - \sqrt{-9}\right)$
52. $\left(2 - \sqrt{-16}\right)\left(1 - \sqrt{-81}\right)$
53. $\left(5 - \sqrt{-4}\right)^2$
54. $\left(3 + \sqrt{-9}\right)^2$
55. $\left(1 - \sqrt{-2}\right)\left(\sqrt{18} - \sqrt{-8}\right)$
56. $\left(\sqrt{2} - \sqrt{-3}\right)\left(\sqrt{8} - \sqrt{-4}\right)$
57. $(2 + i)^3$
58. $(1 - 2i)^3$
59. $\dfrac{17i}{5 - 3i}$
60. $\dfrac{12i}{6 - i}$
61. $\dfrac{i}{1 + i}$
62. $\dfrac{3i}{3 - i}$
63. $\dfrac{2 + 3i}{1 + 2i}$
64. $\dfrac{4 - 3i}{1 + i}$

65. $\dfrac{1-i}{2-3i}$

66. $\dfrac{2+i}{1-9i}$

67. $\dfrac{-2+7i}{-1+i}$

68. $\dfrac{1+4i}{4+i}$

69. $\dfrac{-3-4i}{5+2i}$

70. $\dfrac{3-2i}{-4-11i}$

71. $\dfrac{6+i}{2i}$

72. $\dfrac{2+i}{2i}$

73. $\dfrac{7+\sqrt{-25}}{2-\sqrt{-9}}$

74. $\dfrac{4-\sqrt{-4}}{3+\sqrt{-81}}$

75. $\dfrac{\sqrt{-12}}{\sqrt{-3}+\sqrt{-12}}$

76. $\dfrac{\sqrt{-3}}{\sqrt{-3}+\sqrt{-4}}$

77. $\dfrac{1-\sqrt{-2}}{\sqrt{18}-\sqrt{-8}}$

78. $\dfrac{\sqrt{-3}-1}{\sqrt{-27}-\sqrt{2}}$