8.5 Division of Radicals

In this section we want to talk about how to divide radicals, thereby completing all of our basic operations on radicals.

First we need a property and we need to recall what it means for a radical to be in simplest form.

Quotient Property of Radicals
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers then, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Recall the following from section 8.2.

A radical is in simplest form when:
1. The radicand has no factors that have a power greater than the index. 2. No fractions are underneath the radical. 3. No radicals are in the denominator.

We have already learned how to deal with the first part of this rule. We now need to deal with the second and third parts. It should be clear that to deal with the second part (No fractions are underneath the radical) we simply use the quotient property of radicals stated above. However, to deal with the last part is a little more complicated.

Example 1:

Simplify.

a. $\sqrt[3]{\frac{8}{27}}$

b. $\sqrt[3]{\frac{16x^4}{y^9}}$

Solution:

a. We start by using the quotient property to break the radical over the fraction. Then we simplify numerator and denominator individually as before.

$$\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$$

b. We proceed the same as in part a. Use the quotient property and then simplify.

$$\sqrt[3]{\frac{16x^4}{y^9}} = \frac{\sqrt[3]{16x^4}}{\sqrt[3]{y^9}} = \frac{\sqrt[3]{8x^3(2x)}}{\sqrt[3]{y^9}} = \frac{2x\sqrt[3]{2x}}{y^3}$$

Since there is no radical in the denominator and all the powers are smaller then the index we have fully simplified the radical.

In the example we should notice that the denominators were set up very nicely for us in the sense that the radical completely simplified so that there was no radical left in the denominator.

However, what happens if we have a problem like $\frac{1}{\sqrt{2x}}$. Clearly the radical in the denominator is already simplified. But to simplify the entire fraction we need to find a way to get the radical out of the denominator. To do this we do something called rationalizing the denominator.

To rationalize the denominator we multiply the numerator and denominator by something that will clear the radical from the denominator.

We will need to call upon all that we have learned thus far about radicals to do this rationalizing the denominator.

Example 2:

Simplify. Be sure to rationalize the denominator.

a. $\frac{1}{\sqrt{2x}}$ b. $\sqrt[3]{\frac{27}{4x^2}}$ c. $\sqrt[3]{\frac{16z^3}{y^7}}$ d. $\frac{3u^2}{\sqrt[4]{4u}}$

Solution:

a. To rationalize the denominator we need to multiply the numerator and denominator by whatever it takes to get the radical to drop off. In this case we can use $\sqrt{2x}$. Since we can notice that $\sqrt{2x} \cdot \sqrt{2x} = \sqrt{4x^2} = 2x$. So we proceed as follows

$$\frac{1}{\sqrt{2x}} \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{\sqrt{2x}}{2x}$$

b. Here we need to start by breaking up the radical over the numerator and denominator. Then we should simplify the numerator and denominator separately. We get

$$\sqrt[3]{\frac{27}{4x^2}} = \frac{\sqrt[3]{27}}{\sqrt[3]{4x^2}}$$

$$= \frac{3}{\sqrt[3]{4x^2}}$$

Now we notice that we have a radical in the denominator. Therefore we must rationalize the denominator. The key is multiply numerator and denominator by a radical containing enough to make the exponents on the inside match the index. This works since we know that if the index and the power match then they cancel. So in this case we will need to use $\sqrt[3]{2x}$ since $4 = 2^2$. We get

$$\frac{3}{\sqrt[3]{4x^2}} \frac{\sqrt[3]{2x}}{\sqrt[3]{2x}} = \frac{3\sqrt[3]{2x}}{\sqrt[3]{8x^3}}$$

$$= \frac{3\sqrt[3]{2x}}{2x}$$

Now the radical is completely simplified.

c. Again, we will split the radical to begin the problem. This time, however, we will need to simplify both numerator and denominator before we think about rationalizing the denominator. We proceed as follows.

$$\sqrt[3]{\frac{16z^3}{y^7}} = \frac{\sqrt[3]{16z^3}}{\sqrt[3]{y^7}}$$

$$= \frac{\sqrt[3]{8z^3(2)}}{\sqrt[3]{y^6(y)}}$$

$$= \frac{2z\ \sqrt[3]{2}}{y^2\ \sqrt[3]{y}}$$

Now we need to rationalize the denominator. Remember, we will need to multiply by whatever it takes to get the radicand to have a 3rd power (so that it will match the index). We see clearly we will need a $\sqrt[3]{y^2}$. So we get

$$\frac{2z\ \sqrt[3]{2}}{y^2\ \sqrt[3]{y}}\frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} = \frac{2z\ \sqrt[3]{2y^2}}{y^2\ \sqrt[3]{y^3}}$$

$$= \frac{2z\ \sqrt[3]{2y^2}}{y^3}$$

We can do no more reducing. Therefore the radical is completely simplified.

d. Lastly, we notice that the radical in this example does not simplify. Therefore we will merely have to rationalize the denominator and reduce. Since this time we have an index of 4, we will need to make each value in the radicand have a 4th power. We know that $4 = 2^2$. So we clearly use $\sqrt[4]{4u^3}$. We get

$$\frac{3u^2}{\sqrt[4]{4u}}\frac{\sqrt[4]{4u^3}}{\sqrt[4]{4u^3}} = \frac{3u^2\ \sqrt[4]{4u^3}}{\sqrt[4]{16u^4}}$$

$$= \frac{3u^2\ \sqrt[4]{4u^3}}{2u}$$

$$= \frac{3u\ \sqrt[4]{4u^3}}{2}$$

In the last step we canceled a *u* on top and bottom since they we both outside the radical. We must always make sure we do any canceling that is possible to ensure that we end up in simplest form.

So we see that the key to rationalizing a denominator is multiplying the numerator and denominator by a radical of the same index, whose radicand will make the total radicand have the same powers as the index. This will guarantee that the radical on the denominator will simplify leaving no radicals on the denominator as desired.

What about if we have an expression like $\frac{-3}{2-\sqrt{3}}$? We clearly cannot use the same technique since there are two terms in the denominator and we would have to distribute to both terms.

So we recall the following.

Definition: Conjugates
The expressions $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$ are called <u>conjugates</u>. Conjugates always have a product that is "radical free".

So, since the product of the conjugates always has no radicals, we can use it to rationalize any denominator that has two terms.

We can simply multiply numerator and denominator by the conjugate of the denominator to rationalize the denominator.

<u>Example 3</u>:

Simplify.

a. $\dfrac{-3}{2-\sqrt{3}}$ b. $\dfrac{4}{\sqrt{2a}+\sqrt{2b}}$ c. $\dfrac{3\sqrt{x}-4\sqrt{y}}{3\sqrt{x}+4\sqrt{y}}$

Solution:

a. So as we said above we need to multiply the numerator and denominator by the conjugate of the denominator. The conjugate of the denominator is $2+\sqrt{3}$. So we get

$$\frac{-3}{\left(2-\sqrt{3}\right)}\cdot\frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)}=\frac{-6-3\sqrt{3}}{4+2\sqrt{3}-2\sqrt{3}-3}$$
$$=\frac{-6-3\sqrt{3}}{1}$$
$$=-6-3\sqrt{3}$$

b. Again, we rationalize the denominator by multiplying the numerator and denominator by the conjugate of the denominator, $\sqrt{2a}-\sqrt{2b}$. This gives us

$$\frac{4}{\left(\sqrt{2a}+\sqrt{2b}\right)}\cdot\frac{\left(\sqrt{2a}-\sqrt{2b}\right)}{\left(\sqrt{2a}-\sqrt{2b}\right)}=\frac{4\sqrt{2a}-4\sqrt{2b}}{2a-\sqrt{4ab}+\sqrt{4ab}-2b}$$
$$=\frac{4\sqrt{2a}-4\sqrt{2b}}{2a-2b}$$

Notice that each term in the numerator and denominator has a 2 in common. This means we can factor it out of both and then cancel the two out to completely reduce the expression. We get

$$\frac{4\sqrt{2a}-4\sqrt{2b}}{2a-2b}=\frac{2\left(2\sqrt{2a}-2\sqrt{2b}\right)}{2\left(a-b\right)}$$
$$=\frac{2\sqrt{2a}-2\sqrt{2b}}{a-b}$$

Now that we have no more common factors and all radicals are simplified, the expression is completely simplified.

c. Lastly, we rationalize as we did above.

$$\frac{\left(3\sqrt{x}-4\sqrt{y}\right)\cdot\left(3\sqrt{x}-4\sqrt{y}\right)}{\left(3\sqrt{x}+4\sqrt{y}\right)\cdot\left(3\sqrt{x}-4\sqrt{y}\right)}=\frac{9x-12\sqrt{xy}-12\sqrt{xy}+16y}{9x-12\sqrt{xy}+12\sqrt{xy}-16y}$$

$$=\frac{9x-24\sqrt{xy}+16y}{9x-16y}$$

Since the operation on the numerator and denominator is subtraction, we cannot cancel the 9x or the 16y. Remember, we can only cancel under the operation of multiplication. So we have fully simplified the expression.

8.5 Exercises

Simplify. Be sure to rationalize all denominators.

1. $\sqrt{\frac{4}{9}}$
2. $\sqrt{\frac{1}{16}}$
3. $\sqrt[3]{\frac{27}{64}}$
4. $\sqrt[3]{\frac{125}{8}}$
5. $\sqrt[3]{\frac{1}{27}}$
6. $\sqrt[3]{\frac{64}{216}}$
7. $\sqrt{\frac{18}{y^2}}$
8. $\sqrt{\frac{32}{a^2}}$
9. $\sqrt[3]{\frac{32y^5}{x^3}}$
10. $\sqrt[3]{\frac{54x^4}{y^3}}$
11. $\sqrt[4]{\frac{16b^{11}}{a^8}}$
12. $\sqrt[4]{\frac{48a^9}{b^{12}}}$
13. $\sqrt[5]{\frac{a^8b^{21}}{x^{10}y^{15}}}$
14. $\sqrt[5]{\frac{32}{a^{25}b^5}}$
15. $\sqrt[3]{\frac{y^4z^8}{8x^6}}$
16. $\sqrt{\frac{5c^3}{36a^8b^{12}}}$
17. $\sqrt{\frac{9z^5}{16x^4y^6}}$
18. $\sqrt[3]{\frac{16z^7}{27x^9y^6}}$
19. $\sqrt[3]{\frac{1}{x^3y^9z^{27}}}$
20. $\sqrt[3]{\frac{1}{a^9b^{18}c^3}}$
21. $\sqrt{\frac{8x^2}{y}}$
22. $\sqrt{\frac{9a}{b}}$
23. $\sqrt{\frac{27a^4}{2b^3}}$
24. $\sqrt{\frac{6x^4}{5y^5}}$
25. $\sqrt[3]{\frac{20x^2}{9y^2}}$
26. $\sqrt[3]{\frac{18x}{7y}}$
27. $\sqrt[3]{\frac{8x^4}{3y^7}}$
28. $\sqrt[3]{\frac{54x^5}{2y^{10}}}$
29. $\sqrt[4]{\frac{y^5}{9x}}$
30. $\sqrt[4]{\frac{5}{4y}}$
31. $\sqrt[5]{\frac{2x^2}{3y^3}}$
32. $\sqrt[5]{\frac{15b^3}{4a^2}}$
33. $\frac{\sqrt[3]{12xy}}{\sqrt[3]{3x^2y^2}}$
34. $\frac{\sqrt{8x^2y}}{\sqrt{2xy^2}}$
35. $\frac{5}{\sqrt[3]{9xy^2}}$
36. $\frac{2}{\sqrt[3]{4x^2y^2}}$
37. $\frac{3x}{\sqrt[3]{27x}}$
38. $\frac{4a}{\sqrt[3]{16a^2}}$
39. $\frac{a^3}{\sqrt[4]{ab^2}}$
40. $\frac{x^4}{\sqrt[4]{x^2y^2}}$
41. $\frac{2t^2}{\sqrt[4]{8t^9}}$
42. $\frac{6u^2}{\sqrt[4]{3u}}$
43. $\sqrt{\frac{8ab}{c^4d}}$
44. $\sqrt{\frac{30x^2y}{7z^5}}$

45. $\sqrt[3]{\dfrac{y^8}{81x^4}}$

46. $\sqrt[3]{\dfrac{b^5c^2}{25a^7}}$

47. $\sqrt[4]{\dfrac{16x^9y^5}{z}}$

48. $\sqrt[4]{\dfrac{81ab^{10}}{2x^3}}$

49. $\sqrt[3]{\dfrac{20z^5}{3x^2y^6}}$

50. $\sqrt[3]{\dfrac{30z}{27xy^4}}$

51. $\sqrt[3]{\dfrac{27y^9z^5}{2x}}$

52. $\sqrt[3]{\dfrac{16bc^6}{3a^2}}$

53. $\sqrt[4]{\dfrac{32u^4v^5}{w}}$

54. $\sqrt[4]{\dfrac{273x^2z^5}{8y^9}}$

55. $\sqrt[4]{\dfrac{81c^7}{2a^4b^9}}$

56. $\sqrt[4]{\dfrac{9a^8b^9}{64c^6}}$

57. $\sqrt[4]{\dfrac{1215x^9z^{15}}{2y^6}}$

58. $\sqrt[4]{\dfrac{18x^{14}y^6}{2z^5}}$

59. $\dfrac{6x^2y^3}{\sqrt[7]{32x^2y^3}}$

60. $\dfrac{5xy}{\sqrt[6]{25x^5y^2}}$

61. $\dfrac{7}{\sqrt{2}-3}$

62. $\dfrac{5}{2+\sqrt{2}}$

63. $\dfrac{x}{\sqrt{x}+1}$

64. $\dfrac{2y}{\sqrt{2y}-1}$

65. $\dfrac{1}{\sqrt{2}-\sqrt{3}}$

66. $\dfrac{5}{\sqrt{5}-\sqrt{2}}$

67. $\dfrac{5}{\sqrt{x}-2\sqrt{y}}$

68. $\dfrac{6}{\sqrt{t}+2\sqrt{u}}$

69. $\dfrac{3x}{\sqrt{15}+\sqrt{3}}$

70. $\dfrac{2x}{\sqrt{6}+\sqrt{2}}$

71. $\dfrac{2\sqrt{t}+1}{2\sqrt{t}-1}$

72. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

73. $\dfrac{y+1}{y^2+\sqrt{y}}$

74. $\dfrac{x-2}{\sqrt{x}-x}$

75. $\dfrac{3\sqrt{y}}{2\sqrt{x}-3\sqrt{y}}$

76. $\dfrac{\sqrt{x}}{\sqrt{x}-\sqrt{y}}$

77. $\dfrac{\sqrt{x}+2}{\sqrt{x}-6}$

78. $\dfrac{3+\sqrt{y}}{2-\sqrt{y}}$

79. $\dfrac{2\sqrt{x}-\sqrt{y}}{3\sqrt{x}+5\sqrt{y}}$

80. $\dfrac{\sqrt{u}-\sqrt{v}}{2\sqrt{u}+\sqrt{v}}$