8.2 Simplifying Radicals

In the last section we saw that $16^{1/2} = 4$ since $4^2 = 16$. However, notice that $(-4)^2 = 16$. So 16 has two different square roots. Because of this we need to define what we call the principal square root so that we can distinguish which one we want.

Definition: Principal nth root
The principal nth root of a number is the nth root that has the same sign as the original number. We use radical notation to indicate we the principal nth root.

So by this definition, $\sqrt{16} = 4$ since the 16 is positive and $\sqrt[3]{-27} = -3$ since the 27 is negative.

The idea is this,

- If you have an even index, the principal nth root must be positive (since if n is even, the radicand must be positive or else you don't get a real number)
- If you have an odd index, the principal nth root must have the same sign as the radicand

Now notice the following

$$\sqrt{(-3)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad \sqrt{3^2} = \sqrt{9} = 3$$

Recall for $\sqrt{(-3)^2}$ we would have to follow the order of operations which means evaluating the square first.

The relationship illustrated above motivates the following property.

Property
If n is even, then $\sqrt[n]{a^n} = \lvert a \rvert$. For example, $\sqrt{(-3)^2} = \lvert -3 \rvert = 3$. And If n is odd, then $\sqrt[n]{a^n} = a$. For example, $\sqrt[3]{(-2)^3} = -2$.

In order to simplify the matter, we will always assume that the variables are positive. Therefore we will not need to worry about the absolute value signs. In that case the first part of the property becomes: If n is even, then $\sqrt[n]{a^n} = a$.

So, what this tells us is whenever the index and the power in the radicand match, you end up with just the part inside. That is to say,

When they are the same, the exponent and the radical "cancel".

We use this idea with our rules for rational exponents to simplify radicals. We illustrate with the following example.

Example 1:

Simplify.

a. $-\sqrt{a^6}$ b. $\sqrt[3]{x^3 y^{12}}$ c. $\sqrt[4]{b^{16} c^4}$ d. $\sqrt[5]{-32 x^{15} y^{20}}$

Solution:

a. First notice that the negative is outside the radical and therefore does not cause a problem. The problems only arise when we have a negative under an even indexed radical. So we can simply convert the radical into rational exponents and then simplify as before as follows

$$-\sqrt{a^6} = -\left(a^6\right)^{1/2} = -a^3.$$

b. Again, lets convert the radical notation into exponent notation and simplify accordingly.

$$\begin{aligned}\sqrt[3]{x^3y^{12}} &= \left(x^3y^{12}\right)^{1/3}\\ &= xy^4\end{aligned}$$

c. Likewise we simplify by converting and simplifying.

$$\begin{aligned}\sqrt[4]{b^{16}c^4} &= \left(b^{16}c^4\right)^{1/4}\\ &= b^4c\end{aligned}$$

d. This time we need to just be careful about the numerical part of the radicand. We deal with that the same way we always deal with it. We continue as follows.

$$\begin{aligned}\sqrt[5]{-32x^{15}y^{20}} &= \left(-32x^{15}y^{20}\right)^{1/5}\\ &= \left(-32\right)^{1/5}x^3y^4\\ &= -2x^3y^4\end{aligned}$$

Recall that it is okay to have a negative under an odd index root. It simply means that the number is negative.

This example illustrated how to simplify a radical if the powers underneath are “nice” powers. However, we know that sometimes the powers are not “nice”.

So we need to be able to simplify radical no matter how complicated. For this we need the following.

A radical is in simplest form when:
1. The radicand has no factors that have a power greater than the index. 2. No fractions are underneath the radical. 3. No radicals are in the denominator.

In the rest of this section we want to concentrate on the first one of these rules. That is, the radicand has no factors that have a power greater than the index. The other two rules we will deal with later.

In order to deal with part one of the rule we will need the following property.

Product Property of Radicals
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers then, $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

To illustrate this consider $\sqrt{36}$. We know that this is 6. However, we should be able to get 6 even if we use the property. We can do so as follows

$$\sqrt{36} = \sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$$

This is by no means a proof of the property, merely an example to illustrate its validity.

As we said, we use this property to help simplify radicals.

Example 2:

Simplify.

a. $\sqrt{x^3y^6z^9}$ b. $\sqrt{60xy^7z^{12}}$ c. $\sqrt[3]{-216x^5y^{10}}$ d. $\sqrt[4]{64x^8y^{10}z^{15}}$

Solution:

a. In light of example 1, we should try to find a way to make the powers under the radical become multiples of the index. We know then that they would easily simplify. Since there is no index shown, it is a two. So lets rewrite each variable as having a power that is a multiple of two, times whatever else we need. We do so as follows

$$\sqrt{x^3y^6z^9} = \sqrt{x^2xy^6z^8z}$$

Now we can group together all of the parts that have the "nice" powers in the front and all the extra stuff in the back.

$$\sqrt{x^2xy^6z^8z} = \sqrt{x^2y^6z^8(xz)}$$

Using the product property for radicals we get

$$\sqrt{x^2y^6z^8(xz)} = \sqrt{x^2y^6z^8} \cdot \sqrt{xz}$$

Notice that we can now simplify the front part as we did in example 1.

$$\begin{aligned}\sqrt{x^2y^6z^8} \cdot \sqrt{xz} &= \left(x^2y^6z^8\right)^{1/2} \cdot \sqrt{xz} \\ &= xy^3z^4\sqrt{xz}\end{aligned}$$

All the powers under the radical are smaller than the index and so the radical is simplified.

b. We see then that the object is to write the radicand as having powers that are multiples of the index times whatever is left over. This is the key to simplifying radicals. However, for this example, we have to deal with the 60 as well. To do that we will break the 60 down into its prime factorization and then use the same technique as we did with the variable parts, that is, write as power that are multiples of the index. So, the prime factorization of 60 is $2^2 \cdot 3 \cdot 5$. So we have

$$\sqrt{60xy^7z^{12}} = \sqrt{2^2 \cdot 3 \cdot 5xy^7z^{12}}$$

Since the index is 2 here, we will make the powers multiples of 2 and put the left over parts together at the back of the radical and proceed like above. This gives us

$$\begin{aligned}\sqrt{2^2 \cdot 3 \cdot 5xy^7z^{12}} &= \sqrt{2^2y^6z^{12}(3 \cdot 5xy)} \\ &= \sqrt{2^2y^6z^{12}}\sqrt{3 \cdot 5xy} \\ &= \left(2^2y^6z^{12}\right)^{1/2}\sqrt{15xy} \\ &= 2y^3z^6\sqrt{15xy}\end{aligned}$$

Notice, all powers are smaller than the index. Therefore, the radical is simplified.

c. Again we will start by prime factoring the 216: $216 = 2^3 \cdot 3^3$. Now we continue as before, that is, make everything in the radicand have a power that is a multiple of the index times the left over stuff. We proceed as follows

$$\begin{aligned}\sqrt[3]{-216x^5y^{10}} &= \sqrt[3]{-2^3\cdot 3^3x^3y^9\left(x^2y\right)}\\ &= \sqrt[3]{-2^3\cdot 3^3x^3y^9}\sqrt[3]{x^2y}\\ &= \left(-2^3\cdot 3^3x^3y^9\right)^{1/3}\sqrt[3]{x^2y}\\ &= -2\cdot 3xy^3\sqrt[3]{x^2y}\\ &= -6xy^3\sqrt[3]{x^2y}\end{aligned}$$

Notice, since the index is odd, the negative under the radical can just be carried out since we know that answer will be negative.

d. Lastly, we proceed as we have for all the other examples.

$$\begin{aligned}\sqrt[4]{64x^8y^{10}z^{15}} &= \sqrt[4]{2^6x^8y^{10}z^{15}}\\ &= \sqrt[4]{2^4x^8y^8z^{12}\left(2^2y^2z^3\right)}\\ &= \sqrt[4]{2^4x^8y^8z^{12}}\sqrt[4]{2^2y^2z^3}\\ &= \left(2^4x^8y^8z^{12}\right)^{1/4}\sqrt[4]{4y^2z^3}\\ &= 2x^2y^2z^3\sqrt[4]{4y^2z^3}\end{aligned}$$

So again, the key to the first part of simplifying radicals is to rewrite the powers under the radical as multiples of the index.

Then we simply need to use the product property and properties of rational exponents to finish.

8.2 Exercises

Simplify. Assume all variables represent positive values.

1. $\sqrt{x^2}$
2. $\sqrt[3]{y^3}$
3. $\sqrt[3]{a^9}$
4. $\sqrt{b^4}$
5. $\sqrt{x^2y^2}$
6. $\sqrt{a^4b^2}$
7. $\sqrt[4]{a^8b^8}$
8. $\sqrt[5]{x^{15}y^{10}}$
9. $\sqrt[5]{x^{10}y^{15}z^{20}}$
10. $\sqrt[4]{x^{20}y^{16}z^4}$
11. $\sqrt{4x^4y^2}$
12. $\sqrt{9a^6b^8}$
13. $\sqrt[3]{-27a^9b^6}$
14. $\sqrt[3]{-8x^{12}y^{18}}$
15. $\sqrt[4]{16x^{32}y^{16}}$
16. $\sqrt[4]{81x^8y^{12}}$
17. $\sqrt[4]{81x^{16}y^8z^4}$
18. $\sqrt{49x^8y^6z^{30}}$
19. $\sqrt{144a^6b^{10}c^{14}}$
20. $\sqrt{36a^{24}b^6c^8}$
21. $\sqrt[5]{-32x^5y^{30}z^{45}}$
22. $\sqrt[4]{a^{16}b^{20}c^4}$
23. $\sqrt[4]{256x^{12}y^{36}z^{16}}$
24. $\sqrt[3]{-343x^9y^{24}z^{27}}$
25. $\sqrt{8}$
26. $\sqrt{12}$
27. $\sqrt{18}$
28. $\sqrt{158}$
29. $\sqrt[3]{40}$
30. $\sqrt[3]{16}$
31. $\sqrt[4]{48}$
32. $\sqrt[4]{162}$
33. $\sqrt{54}$
34. $\sqrt{48}$
35. $\sqrt[3]{240}$
36. $\sqrt[3]{144}$
37. $\sqrt{a^3}$
38. $\sqrt[3]{x^5}$
39. $\sqrt[3]{x^8}$
40. $\sqrt{a^5}$
41. $\sqrt{x^3y^5}$
42. $\sqrt{x^7y^4}$
43. $\sqrt[3]{a^6b^{10}}$
44. $\sqrt[3]{x^2y^{14}}$
45. $\sqrt[5]{x^7y^{13}}$
46. $\sqrt[4]{x^9y^{15}}$
47. $\sqrt[5]{x^6y^{12}}$
48. $\sqrt[5]{x^{18}y^{15}}$

49. $\sqrt{8x^9y^{10}}$

50. $\sqrt{27x^5y^3}$

51. $\sqrt[3]{-x^4y^8}$

52. $\sqrt[3]{-40a^8b^{14}}$

53. $\sqrt[4]{48x^{13}y^{15}}$

54. $\sqrt[4]{144x^{10}y^7}$

55. $\sqrt[3]{-100x^{10}y^{10}z^{20}}$

56. $\sqrt[3]{-250x^{10}y^9z^8}$

57. $\sqrt{50xyz^5}$

58. $\sqrt{75x^5y^7z^8}$

59. $\sqrt{60x^4yz^5}$

60. $\sqrt{84x^3y^6z^2}$

61. $\sqrt[5]{-288x^{14}y^{12}z}$

62. $\sqrt[3]{-144x^{13}y^8z^{12}}$

63. $\sqrt[3]{324x^{20}y^{17}z^{18}}$

64. $\sqrt[5]{-32a^{10}b^{30}c^{17}}$

65. $\sqrt[4]{90x^8y^{12}z^4}$

66. $\sqrt[4]{72x^8y^{10}z^4}$

67. $\sqrt[6]{128a^{20}b^{30}c^{17}}$

68. $\sqrt[6]{729a^{16}b^{70}c^{32}}$

69. $\sqrt[7]{x^{63}y^{12}z^{41}}$

70. $\sqrt[7]{x^{49}y^{30}z^{64}}$