8.1 The Principal nth Root

Consider $\left(a^{1/n}\right)^n$.

Using the properties of exponents we would get $\left(a^{1/n}\right)^n = a^{\frac{1}{n}\bullet n} = a^1 = a$.

So the nth power of $a^{1/n}$ is a.

From this we get the following definition.

Definition:
If $a > 0$ and *n* is positive, then $a^{1/n}$ is called the nth root of a. The value of $a^{1/n}$ is a number such that the nth power of the number gives you a.

Example 1:

Evaluate the following.

a. $49^{1/2}$ b. $8^{1/3}$ c. $(-4)^{1/2}$ d. $(-27)^{1/3}$

Solution:

a. We want a number that when we raise it to the second power we get 49. Since $7^2 = 49$, we have $49^{1/2} = 7$.

b. Here we want a number that when raised to the third power will give us 8. Since $2^3 = 8$, we have $8^{1/3} = 2$.

c. This time we need a number that when it is raised to the second power we get a –4. There is no such number since anything to the second power would be a positive. Therefore, we say $(-4)^{1/2}$ is not a real number.

d. Lastly, we want a number that we can raise to the third power and get –27. Since we have an odd power here it is possible to get a negative. Therefore, since $(-3)^3 = 27$, we have $(-27)^{1/3} = -3$.

Now since we are dealing with fractional exponents, we should review some of our basic properties and rules of exponents.

As it turns out, all of the properties and rules that we had for exponents before will still work when the exponent is a rational number.

Rules for Rational Exponents	
1. $a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}$	4. $(a \cdot b)^n = a^n \cdot b^n$; $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
2. $a^n \cdot a^m = a^{n+m}$; $\frac{a^n}{a^m} = a^{n-m}$	5. $a^{-n} = \frac{1}{a^n}$; $\frac{1}{a^{-n}} = a^n$
3. $\left(a^n\right)^m = a^{n \cdot m}$	6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Let us illustrate these rules. Remember that simplifying an expression with exponents means leaving the expression with no negative exponents and also making sure all the values of the same base have been combined.

Example 2:

Simplify. Leave no negative exponents.

a. $16^{3/4}$ b. $a^{1/3} \cdot a^{1/2}$ c. $\left(y^2\right)^{1/4}$ d. $\left(\frac{n}{m}\right)^{3/4}$

e. $x^{-1/2}$ f. $\left(\frac{8}{27}\right)^{-2/3}$ g. $\frac{\left(x^{-5/6} \cdot x^3\right)^{-2/3}}{x^{4/3}}$ h. $\left(\frac{49c^{5/3}}{a^{-1/4}b^{5/6}}\right)^{-3/2}$

Solution:

a. According to property 1 we can write

$$16^{3/4} = \left(16^{1/4}\right)^3$$
$$= 2^3 \qquad 16^{1/4} \textit{ because } 2^4 = 16$$
$$= 8$$

Note, we choose to do the rational exponent inside because it generally will make the values easier to work with. Either way is correct however and would result in an answer of 8.

b. By property number 2, we should add the exponents in this situation. So we get

$$a^{1/3} \cdot a^{1/2} = a^{1/3+1/2}$$

Recall, to add (or subtract) fractions you need to add (or subtract) the numerators over the LCD. So we get

$$a^{1/3+1/2} = a^{5/6}$$

c. According to property 3, we need to multiply the exponents here. So this gives us

$$\left(y^2\right)^{1/4} = y^{2 \cdot 1/4}$$
$$= y^{1/2}$$

d. We use the second part of property 4 here. We can pull the exponent through. When we do this always remember to multiply the value you are pulling though by the exponents that are already there. This gives us

$$\left(\frac{n}{m}\right)^{3/4} = \frac{n^{3/4}}{m^{3/4}}$$

e. For this problem we need to get rid of the negative on the exponent. In order to do that, we need to use property 5. The easiest way to remember how to get rid of a negative exponent is just move the value across the fraction bar. Anytime we do that it changes the value from a positive to a negative exponent. So if we have a negative exponent on the bottom we move it to the top and if its on top (as in our problem) we move it to the bottom. We get

$$x^{-1/2} = \frac{1}{x^{1/2}}$$

Notice, that the only thing that changes with the exponent is the sign. Nothing else about it changes. It is still a ½.

f. The first thing we want to do in this problem is get the negative exponent to a positive one. So we can use property 6 to do that. We simply "flip" the fraction inside and that will change the sign of the exponent. So we will have

$$\left(\frac{8}{27}\right)^{-2/3} = \left(\frac{27}{8}\right)^{2/3}$$

Now we use properties 4 and 1 to completely simplify the expression. We proceed as follows

$$\begin{aligned}\left(\frac{27}{8}\right)^{2/3} &= \frac{27^{2/3}}{8^{2/3}} \\ &= \frac{\left(27^{1/3}\right)^2}{\left(8^{1/3}\right)^2} \\ &= \frac{3^2}{2^2} \\ &= \frac{9}{4}\end{aligned}$$

g. Now we need to use several properties together to simplify the expression. We will begin with pulling the $-2/3$ through the numerator. Then we will combine all the values into one by using both parts of property 2 and get rid of any negative exponents we are left with. We get

$$\frac{\left(x^{-5/6} \cdot x^{3}\right)^{-2/3}}{x^{4/3}} = \frac{x^{-5/6 \bullet -2/3} \cdot x^{3 \bullet -2/3}}{x^{4/3}}$$

$$= \frac{x^{5/9} \cdot x^{-2}}{x^{4/3}}$$

$$= \frac{x^{5/9+(-2)}}{x^{4/3}}$$

$$= \frac{x^{-13/9}}{x^{4/3}}$$

$$= x^{-13/9-4/3}$$

$$= x^{-25/9}$$

$$= \frac{1}{x^{25/9}}$$

h. Lastly, we will need to combine several properties again. We start getting rid of the negative on the $-3/2$ by flipping the fraction inside.

$$\left(\frac{49c^{5/3}}{a^{-1/4}b^{5/6}}\right)^{-3/2} = \left(\frac{a^{-1/4}b^{5/6}}{49c^{5/3}}\right)^{3/2}$$

Next we will pull the exponent though and get rid of the remaining negative exponent. We will then finish any remaining simplifying that needs to be done. We have.

$$\left(\frac{a^{-1/4}b^{5/6}}{49c^{5/3}}\right)^{3/2} = \frac{a^{-1/4 \bullet 3/2}b^{5/6 \bullet 3/2}}{49^{3/2}c^{5/3 \bullet 3/2}}$$

$$= \frac{a^{-3/8}b^{5/4}}{49^{3/2}c^{5/2}}$$

$$= \frac{b^{5/4}}{49^{3/2}a^{3/8}c^{5/2}}$$

$$= \frac{b^{5/4}}{343a^{3/8}c^{5/2}} \quad (49^{3/2} = \left(49^{1/2}\right)^3 = 7^3 = 343)$$

Now that we have a familiarity with the rational exponents we want to see a slightly different but more widely used notation for nth roots.

Definition:
If a is a real number, then $a^{1/n} = \sqrt[n]{a}$, n is called the <u>index</u>, $\sqrt{}$ is called the <u>radical symbol</u> and the expression underneath the radical is called the <u>radicand</u>.

This radical notation is going to be used frequently throughout the rest of this text. Also, rule number 1 above gives us the following very useful rule.

Note: If there is no index on the radical it is assumed to be a two, we call it a square root.

Rule for Radicals: $a^{m/n} = a^{m \bullet 1/n} = \left(a^m\right)^{1/n} = \sqrt[n]{a^m}$ and $a^{m/n} = a^{1/n \bullet m} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m$

So we can see from this that we can easily change between radical and exponent notation. We simply need to remember that the index is the denominator and vice versa.

Example 3:

Rewrite in the alternate notation.

a. $5^{1/2}$ b. $-3a^{2/5}$ c. $\sqrt[5]{4y^7}$ d. $2y\sqrt{x^3}$

Solution:

a. By the above definition, we know that the denominator becomes the index. However, when the index is 2, we need not write it. Therefore, $5^{1/2} = \sqrt{5}$.

b. On this example we need to be careful. Remember that the exponent only goes with the object right before it. In this case that means that the $2/5$ only goes with the a. So the –3 will remain out in front of the radical expression. Also, 5 will be the index, since it is the denominator. So we have $-3a^{2/5} = -3\sqrt[5]{a^2}$.

c. This time we are trying to go back to the exponent notation. Notice that the entire expression $4y^7$ is under the radical symbol. That means that the exponent will go to the entire expression $4y^7$. Since the index becomes the denominator we have $\sqrt[5]{4y^7} = \left(4y^7\right)^{1/5}$.

d. Lastly, we notice that the x^3 is the only part under the radical. Therefore, the 2y will not have the rational exponent on it. So since we see no index, we know it is a 2. Therefore we get $2y\sqrt{x^3} = 2yx^{3/2}$.

8.1 Exercises

Evaluate the following.

1. $4^{1/2}$ 2. $16^{1/2}$ 3. $(-8)^{1/3}$ 4. $(-64)^{1/3}$

5. $1^{1/5}$ 6. $(-1)^{1/7}$ 7. $16^{3/4}$ 8. $81^{3/4}$

9. $(-36)^{3/2}$ 10. $(-9)^{5/2}$ 11. $\left(\frac{1}{32}\right)^{2/5}$ 12. $\left(\frac{25}{9}\right)^{3/2}$

13. $\left(-\frac{8}{125}\right)^{2/3}$

14. $\left(-\frac{1}{216}\right)^{2/3}$

Simplify. Leave no negative exponents.

15. $x^{1/2}\cdot x^{1/4}$

16. $\left(a^{1/3}\right)^{3/4}$

17. $\frac{2a^{1/2}}{a^{1/3}}$

18. $\left(\frac{x}{y}\right)^{2/3}$

19. $\frac{x^{3/4}}{x^{1/2}}$

20. $\left(9x^{1/4}\right)^{1/2}$

21. $\left(b^{2/3}c\right)^{1/2}$

22. $\left(x^{1/2}y\right)^{1/2}$

23. $\frac{5}{x^{-1/4}}$

24. $x^{-1/2}y^{1/4}$

25. $a^{-3/4}b^{-1/2}$

26. $\left(\frac{a^{1/2}}{b^{1/4}}\right)^{8}$

27. $\left(\frac{x^{2/3}}{yz^{1/3}}\right)^{3}$

28. $\frac{a^{-1/2}b^{-1/3}}{c^{2}}$

29. $\frac{x^{-1/3}y}{z^{-1/4}}$

30. $\frac{x^{-2/3}}{y^{1/4}}$

31. $\frac{a^{1/3}b^{1/4}}{a^{-2/3}}$

32. $\frac{3xy}{x^{1/2}y^{-1/2}}$

33. $\frac{x^{1/7}}{x^{1/2}x^{1/3}}$

34. $\left(\frac{2^{-1/2}x^{1/3}}{y^{-1/2}}\right)^{-3}$

35. $\left(27^{-1}a^{-1/2}b^{1/3}\right)^{-1/3}$

36. $\left(32x^{-2/3}y^{-1/4}\right)^{2/5}$

37. $\left(\frac{x^{-1/2}y^{3/4}}{8z^{-1/3}}\right)^{-2/3}$

38. $\left(\frac{abc}{a^{-1/2}b^{1/4}c^{1/2}}\right)^{-2}$

39. $\frac{24a^{-5/3}b^{1/2}}{36a^{-1/3}b^{1/4}}$

40. $\frac{18y^{4/3}z^{-1/3}}{24y^{-2/3}z}$

41. $\left[\left(\frac{3m^{1/6}n^{1/3}}{4n^{-2/3}}\right)^{2}\right]^{-1}$

42. $\left[\left(\frac{x^{1/2}y^{1/3}}{2x^{-1/2}}\right)^{-2}\right]^{-3}$

43. $\left[\frac{6x^{-1/4}y^{3/4}}{\left(z^{-1}\right)^{-1/2}}\right]^{-4}$

44. $\left(\frac{a^{-1/2}b^{-1/2}c^{1/2}}{bc}\right)^{-2}$

45. $\left(\frac{7x^{-4/7}y}{24x^{-1/3}y^{4/3}}\right)^{-1}$

46. $\left(\frac{3x^{-1/2}y}{12x^{1/4}y^{-2/3}}\right)^{-2}$

47. $\left(\frac{4a^{-3/4}b^{1/3}}{28a^{5/4}b^{-1/6}}\right)^{-2}$

48. $\left(\frac{x^{2}y^{-1/3}}{3x^{-1/3}y^{1/2}}\right)^{2}$

49. $\left(3a^{-1/2}b^{1/3}\right)\left(2a^{1/2}b^{-2/3}\right)^{-1}$

50. $\left(4a^{-1/3}b^{1/4}\right)^{1/2}\left(2ab^{-3/4}\right)$

51. $\left[\left(\frac{81a^{-1/2}}{b^{-2/3}}\right)^{-1/2}\right]^{-1/2}$

52. $\left(\frac{4a^{2/3}b^{-2/3}}{25a^{-1}c^{-3/4}}\right)^{-3/2}$

53. $\frac{\left(9xy^{1/2}\right)^{-1/2}}{\left(216x^{1/2}y^{-1}\right)^{-1/3}}$

54. $\frac{36x^{-1/4}y^{-1/5}}{\left(6x^{1/2}y^{1/3}\right)^{2}}$

55. $\left(\dfrac{3x^{1/2}y}{z^{1/3}}\right)^{-1}\left(\dfrac{6x^{1/2}z}{y^{-1/3}}\right)^{2}$

56. $\left(\dfrac{2ab^{-1/2}}{a^{1/2}b}\right)\left(\dfrac{9a^{-1/2}b}{a^{1/2}b^{1/3}}\right)^{-1/2}$

Rewrite in the alternate notation.

57. $2^{1/2}$

58. $3^{1/4}$

59. $x^{2/3}$

60. $y^{3/4}$

61. $2x^{3/2}y$

62. $4ab^{2/3}$

63. $(x+y)^{3/2}$

64. $(3-x)^{2/5}$

65. $\left(12ab^{1/4}\right)^{1/2}$

66. $\left(-6c^{1/2}d\right)^{2/3}$

67. $\sqrt{y^3}$

68. $\sqrt[3]{x}$

69. $2x\sqrt[3]{yz}$

70. $3c\sqrt{ab}$

71. $\sqrt[4]{4a^2b^3}$

72. $\sqrt[5]{25x^3y^5}$

73. $-2x\sqrt{y^3}\,\sqrt[3]{z^4}$

74. $7\sqrt[4]{a^3}\,\sqrt{b^9}$

75. $\sqrt[5]{2x+3y^3}$

76. $\sqrt{x^2+y}$