7.3 Solving Systems by Elimination

In the last section we saw the Substitution Method. It turns out there is another method for solving a system of linear equations that is also very good.

First, we will need a new property for this new method.

Equation Correspondence Property
If $a = b$ and $c = d$ then, $a + c = b + d$.

As we did in the last section, let's start with an example.

Example 1:

Solve $\begin{aligned} x + y &= 2 \\ 3x - y &= 10 \end{aligned}$

Solution:

First, notice that the equations have opposite coefficients for y. Since this is the case, if we use the Equation Correspondence Property to add the corresponding sides together, the y's should cancel.

$$\begin{array}{r} x + y = 2 \\ \underline{3x - y = 10} \\ 4x \quad\; = 12 \end{array}$$

This leaves us with a very easy equation to solve. Dividing by 4 on both sides will give us x = 3

Now we simply need to get the y-value that goes with it. So, just like in the previous section, we just have to plug this value into either original equation and solve. Let's use the first equation.

x + y = 2
3 + y = 2
-3 -3
y = -1

Since the solution to a system is always written as an ordered pair, the solution here must be (3, -1)

The process we used for solving the system in Example 1 is called the Elimination Method, since that is the primary goal, to eliminate one the variables. Here are the general steps for the Elimination Method.

Solving a System by Elimination Method

1. Write the equations in standard form, Ax + By = C.
2. If necessary, multiply one or both equations by a constant that will make the coefficients of one of the variables opposites of each other.
3. Add the corresponding sides of the equations and solve.
4. Substitute the value from step 3 into either original equation and solve.
5. Write the solution as the ordered pair from steps 3 and 4.
6. Check.

Now let's do some examples.

Example 2:

Solve by using the Elimination Method.

a. $\begin{matrix} -2x + 3y = 6 \\ 4x + y = 2 \end{matrix}$ b. $\begin{matrix} 4x + 7y = 11 \\ 8x - 3y = -4 \end{matrix}$ c. $\begin{matrix} 5x - 2y = 8x - 1 \\ 2x + 7y = 4y + 9 \end{matrix}$

Solution:

a. First, we see that neither of the variables has opposite coefficients. This means we will have to multiply one (or possibly) both equations by a constant in order to get one of the variables to have opposite coefficients.

   So, we need to start by deciding which variable to eliminate. Either variable will work, but in this case, it seems like it might be easier to eliminate the x's since they already have opposite signs.

   This means we will have to multiply the top equation by 2, so that we can get a -4x which is opposite of the 4x which is in the bottom equation. This gives

$$\begin{matrix} 2(-2x + 3y = 6) \\ 4x + y = 2 \end{matrix} \longrightarrow \begin{matrix} -4x + 6y = 12 \\ 4x + y = 2 \end{matrix}$$

   Now we can add the sides as we did in Example 1 to eliminate the x's. Then solve.

$$\begin{array}{r} -4x + 6y = 12 \\ \underline{4x + y = 2} \\ \frac{7y}{7} = \frac{14}{7} \\ y = 2 \end{array}$$

   Then substitute y = 2 into either original equation and solve for x. Let's put it into the bottom equation.

$$\begin{array}{r} 4x + y = 2 \\ 4x + 2 = 2 \\ -2 \quad -2 \\ \frac{4x}{4} = \frac{0}{4} \\ x = 0 \end{array}$$

   So the solution to the system is (0, 2)

b. Again, here we need to start by deciding which variable we want to eliminate. In this case, it looks like the x's will be the best, since we would only need to multiply the top equation by -2 to get the coefficients of x to be -8 and 8.

$$\begin{matrix} -2(4x + 7y = 11) \\ 8x - 3y = -4 \end{matrix} \longrightarrow \begin{matrix} -8x - 14y = -22 \\ 8x - 3y = -4 \end{matrix}$$

   Now we add the sides, to eliminate the x's and solve.

$$\begin{array}{r} -8x - 14y = -22 \\ 8x - 3y = -4 \\ \hline \frac{-17y}{-17} = \frac{-26}{-17} \\ y = \frac{26}{17} \end{array}$$

Even though the y value is incredibly inconvenient, we can still get the x that goes with it. We just plug it into one of the original equations and solve. Let's put it into the top equation.

$$\begin{array}{c} 4x + 7y = 11 \\ 4x + 7\left(\frac{26}{17}\right) = 11 \\ 17 \cdot \left(4x + \frac{182}{17}\right) = (11) \cdot 17 \\ 68x + 182 = 187 \\ -182 \quad -182 \\ \frac{68x}{68} = \frac{5}{68} \\ x = \frac{5}{68} \end{array}$$

Clear the fractions by multiplying by the LCD, 17

So the solution is $\left(\frac{5}{68}, \frac{26}{17}\right)$.

c. This time, the equations do not start in standard form. However, getting them to standard form is fairly easy. We simply need to get all the variable terms to the same side.

$$\begin{array}{c} 5x - 2y = 8x - 1 \\ -8x \qquad -8x \end{array} \longrightarrow -3x - 2y = -1$$

$$\begin{array}{c} 2x + 7y = 4y + 9 \\ -4y \quad -4y \end{array} \longrightarrow 2x + 3y = 9$$

Now we have the system $\begin{array}{c} -3x - 2y = -1 \\ 2x + 3y = 9 \end{array}$. So we see that neither of the variables is easy to eliminate. So we can just pick whichever we want to eliminate. Let's eliminate the y's this time.

To do so, we will have to multiply both equations by constants to get the coefficients to cancel. Clearly, we want the coefficients to be the LCM of 2 and 3, that is 6. So we will multiply the top equation by 3 and the bottom by 2.

$$\begin{array}{c} 3(-3x - 2y = -1) \\ 2(2x + 3y = 9) \end{array} \longrightarrow \begin{array}{c} -9x - 6y = -3 \\ 4x + 6y = 18 \end{array}$$

Now, we add the sides and solve.

$$\begin{array}{r} -9x - 6y = -3 \\ 4x + 6y = 18 \\ \hline \frac{-5x}{-5} \qquad = \frac{15}{-5} \\ x = -3 \end{array}$$

Then plug in x = -3 to either original, standard form, equation, let's go with the bottom.

$$
\begin{aligned}
2x + 3y &= 9 \\
2(-3) + 3y &= 9 \\
-6 + 3y &= 9 \\
+6 \qquad &+ 6 \\
\frac{3y}{3} &= \frac{15}{3} \\
y &= 5
\end{aligned}
$$

So the solution is (-3, 5).

Just like always with systems of linear equations, we can have one solution, no solution or infinite solutions. The equations in Example 2 all had one solution. So when using the Elimination Method what happens with the no solution and infinite solution cases. We will see in the next example.

Example 3:

Solve by using the Elimination Method.

a. $\begin{aligned} 10x - 5y &= 7 \\ 2x - y &= 4 \end{aligned}$ b. $\begin{aligned} x &= 2y + 1 \\ 3x - 6y &= 3 \end{aligned}$

Solution:

a. As we did in the previous examples, we need to first decide which variable to eliminate. Clearly, they are both somewhat reasonable to eliminate. So, let's eliminate the x's.

To do so, we need to start by multiplying the bottom equation by -5.

$$
\begin{aligned} 10x - 5y &= 7 \\ -5(2x - y &= 4) \end{aligned} \longrightarrow \begin{aligned} 10x - 5y &= 7 \\ -10x + 5y &= -20 \end{aligned}
$$

Adding corresponding sides gives us

$$
\begin{aligned}
10x - 5y &= 7 \\
\underline{-10x + 5y} &\underline{= -20} \\
0 &= -13
\end{aligned}
$$

Just as we saw in the last section, this must mean the system has no solution.

b. First we need to get the top equation into standard form. This is done by simply subtracting 2y from both sides. So we have

$$
\begin{aligned} x &= 2y + 1 \\ 3x - 6y &= 3 \end{aligned} \longrightarrow \begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned}
$$

Now we decide which variable to eliminate. They seem equally simple, so let's eliminate the y's. So multiply the top equation by -3

$$
\begin{aligned} -3(x - 2y &= 1) \\ 3x - 6y &= 3 \end{aligned} \longrightarrow \begin{aligned} -3x + 6y &= -3 \\ 3x - 6y &= 3 \end{aligned}
$$

Adding sides we have

$$
\begin{aligned}
-3x + 6y &= -3 \\
\underline{3x - 6y} &\underline{= 3} \\
0 &= 0
\end{aligned}
$$

So, as we saw in the last section, this must mean the system has infinite solutions.

One of the biggest challenges in solving systems of equations is determining which method to use. However, either method will work for every system. That being said, sometimes one method is preferred over another.

Usually, if one of the variables is already solved, the substitution method is best. If the equations are set up so that they are already in standard form, however, then usually elimination method is best. Graphing to solve a system is never a good idea. It is far to inaccurate by hand.

Example 4:

Solve using any method.

a. $\begin{aligned} 4x + 5y &= -2 \\ 3x - 2y &= -36 \end{aligned}$ b. $\begin{aligned} x &= 4y + 4 \\ x &= 5y - 10 \end{aligned}$

Solution:

a. Since the equations are already in standard form, the Elimination Method seems like the best choice to solve this system. So we start by eliminating the y terms, by multiplying the top by 2 and the bottom by 5.

$$\begin{matrix} 2(4x + 5y = -2) \\ 5(3x - 2y = -36) \end{matrix} \longrightarrow \begin{matrix} 8x + 10y = -4 \\ 15x - 10y = -180 \end{matrix}$$

Now add the sides and solve

$$\begin{aligned} 8x + 10y &= -4 \\ \underline{15x - 10y} &\underline{= -180} \\ \frac{23x}{23} &= \frac{-184}{23} \\ x &= -8 \end{aligned}$$

Plugging this into the top equation gives us

$$\begin{aligned} 4x + 5y &= -2 \\ 4(-8) + 5y &= -2 \\ -32 + 5y &= -2 \\ +32 \qquad & \quad +32 \\ \frac{5y}{5} &= \frac{30}{5} \\ y &= 6 \end{aligned}$$

So the solution is (-8, 6).

b. This time, it seems as though the Substitution Method would be best since the equations are both solved for x. So since each left side equals x, the right sides must be equal, which is the same as substituting the bottom equation into the top. We have

$$\begin{aligned} 5y - 10 &= 4y + 4 \\ -4y \qquad & \quad -4y \\ y - 10 &= 4 \\ +10 & \quad +10 \\ y &= 14 \end{aligned}$$

Now putting this value into the top equation will give us

$$\begin{aligned} x &= 4y + 4 \\ x &= 4(14) + 4 \\ x &= 56 + 4 \\ x &= 60 \end{aligned}$$

So the solution is (60, 14).

7.3 Exercises

Solve by using the Elimination Method.

1. $\begin{array}{c} -2x + y = 10 \\ 4x - y = -14 \end{array}$

2. $\begin{array}{c} 2x + y = 15 \\ 4x - y = 15 \end{array}$

3. $\begin{array}{c} -x + 9y = -5 \\ x - 5y = 1 \end{array}$

4. $\begin{array}{c} x + 3y = 6 \\ -x + 2y = 9 \end{array}$

5. $\begin{array}{c} x + 3y = 18 \\ -x - 4y = -25 \end{array}$

6. $\begin{array}{c} 4x - y = 10 \\ 4x + y = 6 \end{array}$

7. $\begin{array}{c} -7x - y = 13 \\ 8x + y = -14 \end{array}$

8. $\begin{array}{c} x - y = 0 \\ x + y = 2 \end{array}$

9. $\begin{array}{c} 3x + y = -14 \\ -2x - y = 9 \end{array}$

10. $\begin{array}{c} 4x - 3y = -5 \\ 4x + 4y = 16 \end{array}$

11. $\begin{array}{c} 2x - y = 17 \\ x - y = 10 \end{array}$

12. $\begin{array}{c} x + 5y = 53 \\ x - 5y = -47 \end{array}$

13. $\begin{array}{c} x + 4y = -3 \\ x + 7y = -12 \end{array}$

14. $\begin{array}{c} x + 7y = 24 \\ x - 9y = -24 \end{array}$

15. $\begin{array}{c} 3x + y = -21 \\ x + y = -5 \end{array}$

16. $\begin{array}{c} 3x + 2y = -13 \\ 3x + 4y = 1 \end{array}$

17. $\begin{array}{c} 2x + 14y = -18 \\ x - 9y = 23 \end{array}$

18. $\begin{array}{c} 3x + 7y = 14 \\ 2x + 7y = 21 \end{array}$

19. $\begin{array}{c} x + y = 5 \\ 3x + 3y = 1 \end{array}$

20. $\begin{array}{c} 4x - 4y = -16 \\ x - 2y = -11 \end{array}$

21. $\begin{array}{c} -2x - y = 3 \\ 4x + 2y = -6 \end{array}$

22. $\begin{array}{c} 5x + 5y = 40 \\ 3x + 4y = 24 \end{array}$

23. $\begin{array}{c} 5x - y = 31 \\ 4x + 2y = 8 \end{array}$

24. $\begin{array}{c} 55x - y = -17 \\ 3x - 4y = 17 \end{array}$

25. $\begin{array}{c} x - 2y = 3 \\ 3x + 3y = 9 \end{array}$

26. $\begin{array}{c} 2x + y = 10 \\ 4x + 2y = 2 \end{array}$

27. $\begin{array}{c} 6x + 3y = 13 \\ 9x + 4y = 17 \end{array}$

28. $\begin{array}{c} 5x + 4y = 1 \\ 3x - 3y = -48 \end{array}$

29. $\begin{array}{c} 4x + 3y = -10 \\ 9x + 2y = 6 \end{array}$

30. $\begin{array}{c} x - y = 2 \\ -2x + 2y = -4 \end{array}$

31. $\begin{array}{c} 8x + 7y = -11 \\ 4x - 3y = 14 \end{array}$

32. $\begin{array}{c} -3x + 2y = 23 \\ 5x + 2y = -17 \end{array}$

33. $\begin{array}{c} 4x + 10 = 3y \\ 7x + 3y = -12 \end{array}$

34. $\begin{array}{c} 4x + 9y = -19 \\ -7y = 4x + 13 \end{array}$

35. $\begin{array}{c} 3x = 4y + 17 \\ 8y = 11 - 9x \end{array}$

36. $\begin{array}{c} 2x = 3y + 1 \\ 5x = 5y + 30 \end{array}$

37. $\begin{array}{c} y = -2x - 3 \\ 4x = 19 + 3y \end{array}$

38. $\begin{array}{c} 2x = y + 10 \\ x + 3y = 33 \end{array}$

39. $\begin{array}{c} 2x = 3y + 7 \\ 4x - 6y = 14 \end{array}$

40. $\begin{array}{c} y = -4x + 32 \\ 4y + x = 38 \end{array}$

41. $\begin{array}{c} y = \frac{1}{2}x + 4 \\ x - 2y = 4 \end{array}$

42. $\begin{array}{c} y = -x - 1 \\ y = -\frac{1}{4}x - 19 \end{array}$

43. $-\frac{2}{5}x + \frac{1}{4}y = 3$
$\frac{1}{4}x - \frac{2}{5}y = -3$

44. $\frac{1}{4}x - \frac{1}{3}y = 4$
$\frac{2}{7}x - \frac{1}{7}y = \frac{1}{10}$

45. $\frac{2}{7}x - \frac{1}{5}y = \frac{44}{35}$
$\frac{1}{3}x - \frac{5}{4}y = \frac{7}{2}$

46. $\frac{1}{6}x + \frac{1}{2}y = -2$
$\frac{2}{3}x + \frac{3}{4}y = 2$

Solve by any method.

47. $x + y = 4$
$x - y = 2$

48. $x - 2y = 0$
$2x - y = 6$

49. $x - 2y = 5$
$-3x + 6y = 4$

50. $5x + y = 4$
$5x - 3y = 8$

51. $3x - y = 2$
$6x - 2y = 4$

52. $2x + 3y = 2$
$3x - 2y = 3$

53. $x + 10y = -7$
$-2x + 5y = 4$

54. $5x + 3y = 4$
$2x - y = 5$

55. $\frac{1}{2}x + \frac{1}{3}y = \frac{25}{6}$
$x - y = 5$

56. $\frac{1}{2}x + \frac{1}{4}y = \frac{7}{4}$
$\frac{2}{3}x - \frac{1}{3}y = \frac{13}{3}$