

7.2 Solving Systems by Substitution

As we can see from the examples and exercises in section 7.1, the graphing method for solving a system of equations is less than ideal. This is especially true in cases where the solution ends up being fractional or decimal.

For this reason we want to develop a couple of other methods for solving a system that are much more precise, and much more preferred.

Let's start with a simple example to get us going on our first of these other, more ideal, methods.

Example 1:

Solve
$$\begin{aligned} 2x + 3y &= 9 \\ x &= y - 3 \end{aligned}$$

Solution:

Notice, first, that the bottom equation is solved for x . Since the equation states $x = y - 3$, this means that x and the expression $y - 3$ are equal and therefore interchangeable. So why don't we take the expression $y - 3$ and insert it into the first equation, and see what happens.

$$\begin{array}{ll} 2x + 3y = 9 & \text{Put } y - 3 \text{ in place of } x \\ 2(y - 3) + 3y = 9 & \\ 2y - 6 + 3y = 9 & \text{Combine like terms} \\ 5y - 6 = 9 & \\ +6 & +6 \\ \hline 5y &= 15 \\ \hline 5 &= 5 \\ y &= 3 \end{array}$$

So we get $y = 3$. This is the y -value of the solution to the system. All we need to do then is get the x -value that goes with it. So, as we have always done, let's plug the 3 in for y into one of the original equations and get the x -value. Let's put it into the bottom equation, since it seems easier to do. We have

$$\begin{aligned} x &= y - 3 \\ x &= 3 - 3 \\ x &= 0 \end{aligned}$$

So $x = 0$ and $y = 3$. So, since the solution to a system is always written as an ordered pair, the solution to our system must be $(0, 3)$.

The technique that we just used is called the Substitution Method for solving a system. The good news about the Substitution Method is that it will always work, no matter the linear system, and no matter what the solution looks like.

Solving a System by Substitution Method

1. Solve one equation for one variable.
2. Substitute the expression from step 1 into the other equation and solve.
3. Substitute the value from step 2 into either original equation and solve.
4. The solution is the ordered pair from steps 2 and 3.
5. Check.

Let's look at some more examples.

Example 2:

Solve the systems by using the Substitution Method.

a.
$$\begin{aligned} 3x + y &= 10 \\ 2x + y &= 5 \end{aligned}$$

b.
$$\begin{aligned} x + 5y &= 7 \\ 2x + 3y &= 5 \end{aligned}$$

c.
$$\begin{aligned} -5x + 2y &= -14 \\ 3x + 2y &= 34 \end{aligned}$$

Solution:

- a. We start the substitution method by deciding which equation and which variable to solve for. It's always best to solve for whichever variable appears to be the easiest to get alone on one side.

For this system, it looks like solving either equation for y would be the simplest. So let's just solve the top equation for y .

$$\begin{aligned} 3x + y &= 10 \\ -3x &\quad -3x \\ \hline y &= -3x + 10 \end{aligned}$$

Now, we substitute the expression we just found into the other equation in the system and solve for the remaining variable.

$$\begin{aligned} 2x + y &= 5 \\ 2x + (-3x + 10) &= 5 \\ 2x - 3x + 10 &= 5 \\ -x + 10 &= 5 \\ -10 &\quad -10 \\ \hline -x &= -5 \\ x &= 5 \end{aligned}$$

Now, we substitute $x = 5$ into either one of the original equations and solve for y . It turns out, the best equation to substitute $x = 5$ in, is the equation $y = -3x + 10$. Even though it is not an "original" equation, it is equivalent to an original equation, and it is much easier to use. We have

$$\begin{aligned} y &= -3x + 10 \\ y &= -3(5) + 10 \\ y &= -15 + 10 \\ y &= -5 \end{aligned}$$

So since $x = 5$ and $y = -5$, the solution to the system is $(5, -5)$.

- b. Again, we start by deciding which equation and which variable to solve. In this case, it looks like the first equation for x is probably the simplest variable to get alone. So that is what we will do.

$$\begin{aligned} x + 5y &= 7 \\ -5y &\quad -5y \\ \hline x &= -5y + 7 \end{aligned}$$

Now, we substitute this expression into the other equation for x , since that is what it equals.

$$\begin{aligned}
 2x + 3y &= 5 \\
 2(-5y + 7) + 3y &= 5 \\
 -10y + 14 + 3y &= 5 \\
 -7y + 14 &= 5 \\
 -14 &\quad -14 \\
 \frac{-7y}{-7} &= \frac{-9}{-7} \\
 y &= \frac{9}{7}
 \end{aligned}$$

Now we take this value of y, and plug it into $x = -5y + 7$ to get the x value that goes with it.

$$\begin{aligned}
 x &= -5y + 7 \\
 x &= -5\left(\frac{9}{7}\right) + 7 \\
 x &= -\frac{45}{7} + 7 \\
 x &= -\frac{45}{7} + \frac{49}{7} \\
 x &= \frac{4}{7}
 \end{aligned}$$

So our solution is $\left(\frac{4}{7}, \frac{9}{7}\right)$.

- c. This time, the system does not have a variable that is particularly easy to solve for. So in this situation, we just need to pick a variable, and go for it. Let's take the top equation for y.

$$\begin{aligned}
 -5x + 2y &= -14 \\
 +5x &\quad +5x \\
 \frac{2y}{2} &= \frac{5x}{2} - \frac{14}{2} \\
 y &= \frac{5}{2}x - 7
 \end{aligned}$$

Unfortunately this leaves us with a fraction, but, that would have happened no matter which variable we chose to solve for. So we will substitute it into the other equation just like before.

$$\begin{aligned}
 3x + 2y &= 34 \\
 3x + 2\left(\frac{5}{2}x - 7\right) &= 34 \\
 3x + 5x - 14 &= 34 \\
 8x - 14 &= 34 \\
 +14 &\quad +14 \\
 \frac{8x}{8} &= \frac{48}{8} \\
 x &= 6
 \end{aligned}$$

Now plug $x = 6$ into our equation from step 1 to get

$$\begin{aligned}
 y &= \frac{5}{2}x - 7 \\
 y &= \frac{5}{2}(6) - 7
 \end{aligned}$$

$$\begin{aligned}y &= 15 - 7 \\y &= 8\end{aligned}$$

So the solution to the system is (6, 8).

Recall that the solution to a linear system can be one point, as it was for all of the systems in Example 2, no points, or infinitely many points.

So what happens in the cases of “no solution” and “infinite solutions”. We will take a look in the next example.

Example 3:

Solve the systems by using the Substitution Method.

$$\begin{array}{ll}\text{a. } \begin{aligned}x + y &= 5 \\3x + 3y &= 3\end{aligned} & \text{b. } \begin{aligned}-x + 3y &= -3 \\3x - 9y &= 9\end{aligned}\end{array}$$

Solution:

- a. We will, again, solve for the simplest variable to get alone. It looks like solving the top equation for x or y would be best. Let's solve for x.

$$\begin{aligned}x + y &= 5 \\-y &\quad -y \\x &= -y + 5\end{aligned}$$

Now substitute this into the other equation.

$$\begin{aligned}3x + 3y &= 3 \\3(-y + 5) + 3y &= 3 \\-3y + 15 + 3y &= 3 \\15 &= 3\end{aligned}$$

Since the variables canceled out, we end up with a statement that is complete nonsense (15 does not actually equal 3). Therefore, we can conclude that this system must have no solution.

- b. In this case, none of the variables are easy to get alone. However, it seems like the x in the first equation might be the best choice.

$$\begin{aligned}-x + 3y &= -3 \\-3y &\quad -3y \\-x &\quad -3y \quad 3 \\-1 &= \frac{-3y}{-1} - \frac{3}{-1} \\x &= 3y + 3\end{aligned}$$

So now substitute this into the other equation.

$$\begin{aligned}3x - 9y &= 9 \\3(3y + 3) - 9y &= 9 \\9y + 9 - 9y &= 9 \\9 &= 9\end{aligned}$$

So, as in part a, the variables cancelled out. However, in this case, though, it leaves us with a statement that is clearly a true statement (9 does equal 9). Therefore we can conclude that the system has infinite solutions.

7.2 Exercises

Solve the systems by using the Substitution Method.

1. $y = -5x - 15$
 $-4x = 21 - y$

2. $4y + 5x = 12$
 $y = -3x + 17$

3. $-3x = 4y + 37$
 $x = -4y - 23$

4. $x = -y + 17$
 $x + 5y = 53$

5. $x = 5y + 23$
 $-5x - 3y = 25$

6. $-4y - 2x = -24$
 $y = -2x + 18$

7. $-3y + x = -21$
 $y = -2x - 14$

8. $-2y + 5x = 31$
 $y = -4x + 43$

9. $y = -4x + 12$
 $-5x = 6 - 4y$

10. $y = 2x - 8$
 $2x = -12 - y$

11. $x = y + 5$
 $2x - 2y = 1$

12. $4x - 2y = 6$
 $y = 2x - 3$

13. $x = 5y - 20$
 $x + 2y = 8$

14. $x = -3y + 33$
 $5x - 5y = -15$

15. $x - y = 0$
 $2y = 2x$

16. $4x + 2y = 5$
 $y = -2x$

17. $7x - 4y = -7$
 $5x + y = 22$

18. $-2x + y = -7$
 $4x + 2y = 42$

19. $x - 2y = 15$
 $2x + 3y = 2$

20. $2x + 3y = 48$
 $5x + y = 55$

21. $6x + y = 15$
 $7x + 2y = 10$

22. $y - 3x = -5$
 $y = -5x - 5$

23. $5x + 2y = 21$
 $x + y = 9$

24. $-y + x = 7$
 $2x + y = -4$

25. $7x + y = 4$
 $-6x + 5y = -21$

26. $2x - y = 3$
 $-6x + 3y = -9$

27. $x + 10y = -14$
 $3x + 5y = -22$

28. $-5x + y = -42$
 $4x + 4y = 72$

29. $2x + 3y = 2$
 $-8x - y = 25$

30. $-4x + 4y = -28$
 $x - 2y = 4$

31. $-x + 5y = -16$
 $-3x + 7y = -8$

32. $3x + 3y = 9$
 $-x - 2y = -3$

33. $-x + 2y = 4$
 $2x - 4y = -8$

34. $-x = -y$
 $x = -4y$

35. $-4x + 2y = 9$
 $2x - y = 7$

36. $-x + 2y = 11$
 $5x + y = 55$

37. $-5x + 9y = -12$
 $3x + 2y = 22$

38. $2x + 4y = 8$
 $2x + 2y = 4$

39. $6x + 7y = -9$
 $4x + 5y = -5$

40. $2x + 4y = 22$
 $5x - 5y = 30$

41. $4x + 3y = -13$
 $5x - 2y = 24$

42. $2x + 2y = 9$
 $3x + 3y = 2$

43. $5x - 8y = -4$
 $-10x + 4y = 5$

44. $2x + 3y = -1$
 $3x + 4y = -4$

45. $4x + 9y = -12$
 $2x + 6y = -7$

46. $9x - 2y = -21$
 $2x + 3y = 16$

47. $-\frac{2}{5}x + \frac{1}{4}y = 3$
 $\frac{1}{4}x - \frac{2}{5}y = -3$

48. $\frac{1}{4}x - \frac{1}{3}y = 4$
 $\frac{2}{7}x - \frac{1}{7}y = \frac{1}{10}$

49. $\frac{2}{7}x - \frac{1}{5}y = \frac{44}{35}$
 $\frac{1}{3}x - \frac{5}{4}y = \frac{7}{2}$

50. $\frac{1}{6}x + \frac{1}{2}y = -2$
 $\frac{2}{3}x + \frac{3}{4}y = 2$