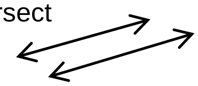
6.6 Parallel and Perpendicular Lines

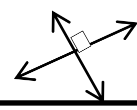
Now that we have talked about lines in general, we want to talk about a couple of ways that lines can interact with each other.

Definition:

Parallel lines- Two lines that never intersect



Perpendicular lines- Two lines that intersect at a right (90^0) angle



The most important thing you need to know about parallel and perpendicular lines is that the relationship between parallel lines is a relationship between the slope of the lines, and the same goes for perpendicular lines.

Slopes of Parallel Lines

Two lines are parallel if and only if they have the **exact same** slope.

For example if the slope of one of the lines is -2, then the slope of the other one has to be -2.

This property is fairly easy to understand why it is true. Remember that the slope of a line represents the steepness of the line. So, if two lines are parallel, they would have to have the same steepness, otherwise they would eventually intersect, making them no longer parallel by definition.

On the other hand, perpendicular lines are a bit more complicated.

Slopes of Perpendicular Lines

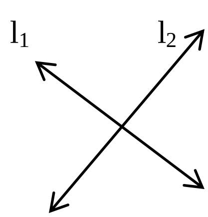
Two lines are perpendicular if and only if their slopes are **opposite-reciprocals**.

That means, to go from the slope of one line to its perpendicular line, you need to change the sign (opposite) and flip it upside down (reciprocal).

For example, if the slope of one of the lines is -2, then the other line must have a slope of ½ or if a line has a slope of $\frac{5}{8}$ then the perpendicular slope would be $-\frac{8}{5}$.

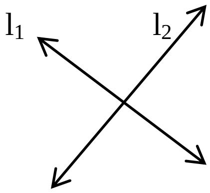
The reason for this relationship is a little involved, but the rough idea can be explained with looking at how the slopes of two perpendicular lines would change if you alter the slope of one of the lines.

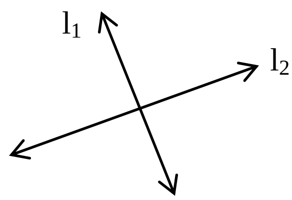
For example, say we had the following perpendicular lines



The first thing we should notice is that clearly is that l_2 has a positive slope and yet l_1 has a negative slope. That is why we have the "opposite" part of the opposite-reciprocals.

Next, let's say we "flatten" out l_2, thereby making its slope smaller. What happens then is, in order to keep the lines perpendicular, we would have to make l_1 more steep, that is, making the slope larger.





This idea of one value increasing while another is decreasing is a reciprocal relationship.

So, this is why the slope relationship between perpendicular lines is opposite-reciprocal.

Let's begin with a couple of simple examples to get these ideas down.

<u>Example 1</u>:

a. Is y = -2x - 3 parallel to y = -2x + 3? b. Is 5x – 3y = 6 perpendicular to 3x + 5y = 2?

Solution:

a. If figure out if two lines are parallel, we need to find and compare the slopes of the lines. The easiest way to find the slope of a line, when we are given the equation of the line, is to put the equation into slope intercept form and simply read the slope off.

   Notice, y = -2x - 3 and y = -2x + 3 are both already in slope intercept form. So clearly, the slope of the first line is -2 and the slope of the second line is also -2.

   Since the slopes are the same, the lines must be parallel.

b. Just like in part a, the best way to determine if the lines are perpendicular is to put the equations in slope intercept form and read off the slopes. So we proceed as follows

$$5x - 3y = 6$$
$$-5x \qquad -5x$$
$$\frac{-3y}{-3} = \frac{-5x}{-3} + \frac{6}{-3}$$
$$y = \frac{5}{3}x - 2$$

$$3x + 5y = 2$$
$$-3x \qquad -3x$$
$$\frac{5y}{5} = \frac{-3x}{5} + \frac{2}{5}$$
$$y = -\frac{3}{5}x + \frac{2}{5}$$

   So the slope of the first line is $\frac{5}{3}$ and the slope of the second line is $-\frac{3}{5}$. Therefore, since the slopes have opposite signs and are reciprocals, the lines must be perpendicular.

<u>Example 2</u>:

Are 2x – 4y = 3 and 2x + 4y = -3 parallel, perpendicular or neither?

Solution:

As in Example 1, we need to determine the slopes of the given lines in order to determine if they are parallel, perpendicular or neither. So we will get the equations into slope-intercept form.

$$2x - 4y = 3$$
$$-2x \qquad -2x$$
$$\frac{-4y}{-4} = \frac{-2x}{-4} + \frac{3}{-4}$$
$$y = \frac{1}{2}x - \frac{3}{4}$$

$$2x + 4y = -3$$
$$-2x \qquad -2x$$
$$\frac{4y}{4} = \frac{-2x}{4} - \frac{3}{4}$$
$$y = -\frac{1}{2}x - \frac{3}{4}$$

So the slope of the first line is $\frac{1}{2}$ and the slope of the second line is $-\frac{1}{2}$. Since the slopes are not exactly the same, the lines are not parallel. Also, even though the slopes are opposites, they are not reciprocals. Therefore, the lines are also not perpendicular.

This means the lines are neither parallel nor perpendicular. So the answer is neither.

Now that we have a sense of how the slopes of parallel and perpendicular lines are related, let's try some more challenging examples which combine the ideas of 6.5 with parallel and perpendicular lines.

Example 3:

Find the equation of the line containing (3, 2) and parallel to 3x + y = -3.

Solution:

In this example, we are given a line and a point upon which we want to construct a line parallel to the given line. As always in parallel lines, this means that the line must have the same slope as the given line. So, we will begin by finding the slope of the line given by getting it into slope-intercept form.

$$3x + y = -3$$
$$-3x \qquad -3x$$
$$y = -3x - 3$$

So, the slope of the line is -3. Since we want the line parallel, then the slope of the line we are trying to find is also -3.

Just like we did in section 6.5, we can label the point (3, 2) as (x_1, y_1) and use the point-slope form to find our equation.

$$y - y_1 = m\,(x - x_1)$$
$$y - 2 = -3(x - 3)$$
$$y - 2 = -3x + 9$$
$$+3x \qquad +3x$$
$$3x + y - 2 = 9$$
$$+2 \quad +2$$
$$3x + y = 11$$

So the line parallel has an equation 3x + y = 11.

Example 4:

Find the equation of the line perpendicular to 2x + 4y = -1 containing (-1,3).

Solution:

Just like we did in Example 3, we will start by finding the slope of the line that we are given.

$$
\begin{aligned}
2x + 4y &= -1 \\
-2x \quad &\quad -2x \\
\frac{4y}{4} &= \frac{-2x}{4} - \frac{1}{4} \\
y &= -\frac{1}{2}x - \frac{1}{4}
\end{aligned}
$$

So the slope is – ½ . Since, this time, we are looking for the line perpendicular, we need to change the sign and flip upside down the slope of the given line. That means the slope we want to use is 2 (opposite-reciprocal of – ½).

Now we simply use the point-slope form as we did in Example 3.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 3 &= 2(x - (-1)) \\
y - 3 &= 2x + 2 \\
-y \quad &\quad -y \\
-3 &= 2x - y + 2 \\
-2 \quad &\quad\quad -2 \\
2x - y &= -5
\end{aligned}
$$

So the line perpendicular has an equation 2x - y = - 5.

Example 5:

Find the equation of the line perpendicular to 4x – y = 8 with y-intercept of -1.

Solution:

Lastly, just like the other examples, we will begin by getting the slope of the given line.

$$
\begin{aligned}
4x - y &= 8 \\
-4x \quad &\quad -4x \\
\frac{-y}{-1} &= \frac{-4x}{-1} + \frac{8}{-1} \\
y &= 4x - 8
\end{aligned}
$$

So the slope of the given line is 4. As in Example 4, we want the line perpendicular. This means we want the opposite-reciprocal of 4. So we change the sign and flip it upside down to get – ¼ . So the slope we will use is - ¼ .

Notice, this time they did not give us a point for use in the point-slope form. However, we know that the y-intercept of -1 means that the graph must go through the point (0, -1), So we can use this with our slope to find the equation as follows.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (-1) &= -\frac{1}{4}(x - 0) \\
4(y + 1) &= 4\left(-\frac{1}{4}x\right) \\
4y + 4 &= -x \\
+x \quad &\quad +x \\
x + 4y + 4 &= 0 \\
-4 \quad &\quad -4 \\
x + 4y &= -4
\end{aligned}
$$

So the equation of the line perpendicular is x + 4y = -4.

6.6 Exercises

Determine if the two lines are parallel.

1. $y = -\frac{4}{5}x + 2$
   $y = \frac{4}{5}x - 3$

2. $y = \frac{2}{3}x - 3$
   $y = \frac{2}{3}x + 1$

3. $2x + 3y = 9$
   $-4x - 6y = 12$

4. $3x + 4y = 8$
   $-12x - 9y = 27$

5. $5x + y = 4$
   $-15x + 3y = 9$

6. $2x - 3y = 6$
   $6x - 9y = -9$

7. The line passing through the points $(2,-5)$ and $(6,2)$.
   The line passing through the points $(4,-3)$ and $(8,4)$.

8. The line passing through the points $(7,3)$ and $(6,1)$
   The line passing through the points $(2,-1)$ and $(1,-3)$.

9. The line passing through the points $(5,-2)$ and $(-3,-4)$.
   The line passing through the points $(3,-4)$ and $(7,-3)$.

10. The line passing through the points $(-2,1)$ and $(4,3)$.
    The line passing through the points $(3,-2)$ and $(5,-1)$.

Determine if the two given lines are perpendicular.

11. $y = -\frac{1}{4}x + 2$
    $y = 4x - 3$

12. $y = \frac{1}{3}x + 7$
    $y = 3x - 3$

13. $5x - y = 7$
    $2x - 10y = 20$

14. $3x - 2y = 4$
    $6x + 9y = 18$

15. $4x - 5y = 10$
    $20x + 16y = 0$

16. $x - 3y = 6$
    $3x - y = 2$

17. The line passing through the points $(4,-3)$ and $(6,2)$.
    The line passing through the points $(3,-2)$ and $(13,2)$.

18. The line passing through the points $(2,7)$ and $(3,-1)$.
    The line passing through the points $(6,-4)$ and $(-2,-5)$.

19. The line passing through the points $(-1,-4)$ and $(2,2)$.
    The line passing through the points $(4,3)$ and $(2,4)$.

20. The line passing through the points $(7,3)$ and $(1,3)$.
    The line passing through the points $(5,2)$ and $(5,-3)$.

Are the two given lines are parallel, perpendicular, or neither?

21. $y = \frac{3}{5}x + 2$
$y = -\frac{5}{3}x - 1$

22. $y = 2x - 3$
$y = 2x + 1$

23. $2y = 7x + 2$
$7y = 2x - 3$

24. $y = 5x + 2$
$5y = -x + 20$

25. $2x + 3y = 9$
$4x + 6y = 12$

26. $3x - 5y = 7$
$10x + 6y = 12$

27. $4x + 4y = 18$
$3x - 2y = 4$

28. $0.7x + 0.1y = 500$
$y = -\frac{1}{7}x - 3$

29. The line passing through the points $(7,6)$ and $(4,3)$.
The line passing through the points $(4,-5)$ and $(2,-7)$.

30. The line passing through the points $(5,-2)$ and $(3,-7)$.
The line passing through the points $(-3,5)$ and $(2,3)$.

31. Find the equation, in standard form, of the line parallel to $y = \frac{2}{3}x - 1$ and passing through the point $(3,-1)$.

32. Find the equation, in standard form, of the line parallel to $y = -\frac{1}{5}x + 7$ and passing through the point $(-10,-3)$.

33. Find the equation, in standard form, of the line perpendicular to $y = -\frac{5}{7}x + 3$ and passing through the point $(15,2)$.

34. Find the equation, in standard form, of the line perpendicular to $y = \frac{3}{4}x - 11$ and passing through the point $(9,2)$.

35. Find the equation, in standard form, of the line passing through the point $(12,-1)$ and perpendicular to $3x - 5y = 25$.

36. Find the equation, in standard form, of the line passing through the point $(0,-3)$ and parallel to $x - 5y = 75$.

37. Find the equation, in standard form, of the line passing through the point $\left(-\frac{2}{3},7\right)$ and parallel to $3x - y = 6$.

38. Find the equation, in standard form, of the line passing through the point $\left(\frac{1}{2},3\right)$ and perpendicular to $x - 2y = 16$.

39. Find the equation, in standard form, of the line passing through the point $\left(\frac{1}{3},-\frac{2}{5}\right)$ and perpendicular to $x = -2$.

40. Find the equation, in standard form, of the line passing through the point $\left(\frac{7}{5},-\frac{2}{5}\right)$ and perpendicular to $y = 3$.

41. Find the equation, in standard form, of the line parallel to $y = 1.3x + 1$ and passing through the point $(-5,-1)$.

42. Find the equation, in standard form, of the line perpendicular to $y = 1.5x - 2$ and passing through the point $(-6,3)$.

43. Find the equation, in standard form, of the line perpendicular to $y = -1.2x + 3$ and passing through the point $(5,-1)$.

44. Find the equation, in standard form, of the line parallel to $y = 3.5x - 7$ and passing through the point $(7,3)$.

45. Find the equation, in standard form, of the line perpendicular to $5x - 7y = 14$ and passing through the point $(-1,7)$.

46. Find the equation, in standard form, of the line parallel to $3x - 4y = 12$ and passing through the point $(5,-2)$.

47. Find the equation, in standard form, of the line parallel to $y = 2x - 5$ with the y -intercept as $y = \frac{2}{3}x + 7$.

48. Find the equation, in standard form, of the line parallel to $x - 3y = 6$ with the same y-intercept as $y = \frac{1}{5}x - 11$.

49. Find the equation, in standard form, of the line perpendicular to $3x-5y=4$ with the same x-intercept as $2x-3y=6$.

50. Find the equation, in standard form, of the line perpendicular to $2x-7y=1$ with the same x-intercept as $3x+5y=15$.

51. Find the equation, in standard form, of the line perpendicular to and with the same x-intercept as 5x – 3y = 15.

52. Find the equation, in standard form, of the line perpendicular to and with the same x-intercept as 3x + 4y = 12.

53. Find the equation, in standard form, of the line that is perpendicular to and with the same y-intercept as a line with x-intercept of (-1, 0) and y-intercept of (0, 2).

54. Find the equation, in standard form, of the line that is parallel to the line with x-intercept of (-4, 0) and y-intercept of (0, -3), and has a y-intercept of ¼.

55. Is the triangle with vertices $(-2,3)$, $(3,3)$, and $(2,1)$ a right triangle?

56. Is the triangle with vertices $(-1,-1)$, $(4,1)$, and $(2,4)$ a right triangle?