

## 5.6 Rational Equations

Now that we have a good handle on all of the various operations on rational expressions, we want to turn our attention to solving equations that contain rational expressions.

The good news is, we will solve these just like we solve any equation containing fractions. That is, we will clear the fractions at some point very early in the solving process. Also, the way we clear fractions is the same as always, by multiplying both sides by the LCD.

Here is the process

### **Solving a Rational Equation**

1. Find the LCD of all the rational expressions in the equation.
2. Multiply both sides by the LCD.
3. Solve the resulting equation.
4. You must check your solutions and throw out any that make the denominator zero.

Notice that we emphasized the “must” in step 4. The reason for this is simple. If you recall from section 5.1, we discussed something called the domain of an expression. Since you can never have a denominator of zero in a fraction, we have to make sure that none of our solutions make our denominator zero.

So, for this reason, when we “check” our answer (in step 4), we don’t have to do a full check to see if the solution works. All we really need to check is to see if any of the solutions make the denominator equal to zero. Any solution that does cannot be included in the solution set.

We call these “extra” solutions, **extraneous solutions**. If it turns out that all of the solutions of a rational equation are extraneous, then the equation would have no solution, meaning, there is no value for the variable that would make the equation a true statement.

Let’s look at some examples.

#### Example 1:

Solve.

a.  $\frac{4}{x} - \frac{7}{3x} = \frac{2}{5}$

b.  $\frac{5}{2x-1} - \frac{7}{3x+2} = \frac{9}{6x^2+x-2}$

c.  $\frac{6}{2p-4} + \frac{4}{p-2} = \frac{1}{p^2-4}$

Solution:

- a. To solve this equation, we will follow the steps outlined above. This means first we need to find the LCD. Clearly, the LCD is  $15x$ . Now, multiply both sides by the LCD, as follows.

$$\frac{4}{x} - \frac{7}{3x} = \frac{2}{5}$$

$$15x \cdot \left( \frac{4}{x} - \frac{7}{3x} \right) = \left( \frac{2}{5} \right) \cdot 15x$$

Now, at this point, we distribute the LCD through to each numerator on the inside, quite similar to how we did it in the last section. This will produce “multiplication pockets” just like it did in the previous section.

$$15x \cdot \left( \frac{4}{x} - \frac{7}{3x} \right) = \left( \frac{2}{5} \right) \cdot 15x$$

$$\frac{60x}{x} - \frac{105x}{3x} = \frac{30x}{5}$$

Now we reduce within our “pockets” and solve what is left.

$$\frac{60\cancel{x}}{\cancel{x}} - \frac{105\cancel{x}}{\cancel{3x}} = \frac{30\cancel{x}}{\cancel{5}}$$

$$60 - 35 = 6x \quad \text{Combine like terms}$$

$$\frac{25}{6} = \frac{6x}{6} \quad \text{Divide by 6}$$

$$x = \frac{25}{6}$$

Lastly, we need to check. Again, we only need to make sure that this value doesn't make any of the denominators zero. Clearly, if we insert  $\frac{25}{6}$  into the original equation, none of the denominators would be zero. Therefore, our solution set is  $\left\{ \frac{25}{6} \right\}$ .

- b. Again we need to start by finding the LCD. To do so we will need to factor all the denominators.

$$\frac{5}{2x-1} - \frac{7}{3x+2} = \frac{9}{6x^2+x-2}$$

$$\frac{5}{2x-1} - \frac{7}{3x+2} = \frac{9}{(2x-1)(3x+2)}$$

So, our LCD is  $(2x-1)(3x+2)$ . Now multiply both sides by this LCD. That is, distribute the LCD to each term on both sides of the equation. Then cancel in our “pockets”.

$$(2x-1)(3x+2) \cdot \left( \frac{5}{2x-1} - \frac{7}{3x+2} \right) = \left( \frac{9}{(2x-1)(3x+2)} \right) \cdot (2x-1)(3x+2)$$

$$\frac{5(2x-1)(3x+2)}{2x-1} - \frac{7(2x-1)(3x+2)}{3x+2} = \frac{9(2x-1)(3x+2)}{(2x-1)(3x+2)} \quad \text{Canceling common factors}$$

$$5(3x+1) - 7(2x-1) = 9$$

Now we just multiply it out and solve what is left.

$$5(3x+1) - 7(2x-1) = 9$$

$$15x + 5 - 14x + 7 = 9 \quad \text{Combine like terms}$$

$$x + 12 = 9$$

$$\begin{array}{r} -12 \\ -12 \end{array}$$

$$x = -3$$

Finally we need to check to make sure this doesn't give us a zero in any denominator. We can see, if we put -3 into all of the (factored) denominators, none of them turn out to be zero. Therefore, the solution set is  $\{-3\}$ .

c. Once again, we need to start by factoring the denominators to find the LCD.

$$\frac{6}{2p-4} + \frac{4}{p-2} = \frac{1}{p^2-4}$$

$$\frac{6}{2(p-2)} + \frac{4}{p-2} = \frac{1}{(p-2)(p+2)}$$

Before we get carried away with the LCD, notice that the front term can be reduced. This will not only keep the equation simplified, but, more importantly, the LCD will be a little simpler. This gives us

$$\frac{6}{2(p-2)} + \frac{4}{p-2} = \frac{1}{(p-2)(p+2)}$$

$$\frac{3}{p-2} + \frac{4}{p-2} = \frac{1}{(p-2)(p+2)}$$

So, our LCD is  $(p-2)(p+2)$ . Multiply it on both sides and reduce the "pockets".

$$(p-2)(p+2) \cdot \left( \frac{3}{p-2} + \frac{4}{p-2} \right) = \left( \frac{1}{(p-2)(p+2)} \right) \cdot (p-2)(p+2)$$

$$\frac{3\cancel{(p-2)}(p+2)}{\cancel{p-2}} + \frac{4\cancel{(p-2)}(p+2)}{\cancel{p-2}} = \frac{1\cancel{(p-2)}(p+2)}{\cancel{(p-2)}(p+2)}$$

$$3(p+2) + 4(p+2) = 1$$

Now distribute and solve the resulting equation.

$$3(p+2) + 4(p+2) = 1 \quad \text{Distribute}$$

$$3p + 6 + 4p + 8 = 1 \quad \text{Combine like terms}$$

$$7p + 14 = 1 \quad \text{Subtract 14}$$

$$\phantom{7p} - 14 \quad \phantom{14} - 14$$

$$\frac{7p}{7} = \frac{-13}{7} \quad \text{Divide by 7}$$

$$p = -\frac{13}{7}$$

When we plug this value back into the original equations denominators, we can see, again, that we don't end up with any zeros. Therefore, the solution is  $\left\{-\frac{13}{7}\right\}$ .

Now, let's look at some examples which are a little harder, and where we have to deal with the solutions not checking.

Example 2:

Solve.

a.  $\frac{x}{x+4} = 3 - \frac{4}{x+4}$

b.  $x + \frac{6}{x-3} = \frac{2x}{x-3}$

c.  $\frac{5}{y-1} + \frac{3}{y-3} = \frac{8}{y-2}$

Solution:

- a. Just like in example 1, we need to start by finding the LCD. Obviously, the LCD here is  $x + 4$ . So multiply both sides by the LCD, reduce and solve what is left.

$$\frac{x}{x+4} = 3 - \frac{4}{x+4} \quad \text{Multiply by the LCD}$$

$$x + 4 \cdot \left(\frac{x}{x+4}\right) = \left(3 - \frac{4}{x+4}\right) \cdot x + 4 \quad \text{Distribute the LCD to each term on both sides}$$

$$\frac{x(x+4)}{x+4} = 3(x+4) - \frac{4(x+4)}{x+4} \quad \text{Cancel in the "pockets"}$$

$$x = 3(x+4) - 4 \quad \text{Distribute}$$

$$x = 3x + 12 - 4 \quad \text{Combine like terms}$$

$$x = 3x + 8 \quad \text{Subtract 3x}$$
$$-3x \quad -3x$$

$$\frac{-2x}{-2} = \frac{8}{-2} \quad \text{Divide by -2 to get x alone}$$

$$x = -4$$

Now, we need to check this in the denominators. If we put -4 into any one of the denominators of the original equation, we would get  $x + 4 = (-4) + 4$ , which equals 0. Since we can't have a zero in a denominator, we have to throw out the solution of -4. That is, -4 is an extraneous solution.

Since there are no other values, the equation has no solution. This means, there is no value of  $x$  that would satisfy the equation. So we simply state: No Solution.

- b. Again, the LCD here is very easy to see. It is  $x - 3$ . So we proceed as we did with the other examples as follows.

$$x + \frac{6}{x-3} = \frac{2x}{x-3} \quad \text{Multiply by the LCD}$$

$$x - 3 \cdot \left(x + \frac{6}{x-3}\right) = \left(\frac{2x}{x-3}\right) \cdot x - 3 \quad \text{Distribute } x - 3 \text{ to each term}$$

$$x(x-3) + \frac{6(x-3)}{x-3} = \frac{2x(x-3)}{x-3} \quad \text{Cancel in the "pockets" and multiply everything else out}$$

$$x^2 - 3x + 6 = 2x$$

As we saw in chapter 4, we have an equation containing a power larger than 1. Therefore, we have to solve this equation by factoring. So we get everything on one side, factor and set each factor equal to zero.

$$\begin{array}{r} x^2 - 3x + 6 = 2x \\ -2x \quad -2x \end{array} \quad \text{Move everything to one side}$$

$$x^2 - 5x + 6 = 0 \quad \text{Factor}$$

$$(x - 2)(x - 3) = 0$$

$$\begin{array}{r} x - 2 = 0 \quad x - 3 = 0 \\ +2 \quad +2 \quad +3 \quad +3 \\ x = 2 \quad x = 3 \end{array} \quad \text{Set each factor to zero and solve}$$

This time we have to check two answers. Again, we need to throw out any answer that makes a denominator of the original equation zero.

Putting the 2 into the original denominators gives  $x - 3 = (2) - 3 = -1$ . Since it is not zero, 2 is a good solution and we get to keep it.

Putting 3 into the denominators gives  $x - 3 = (3) - 3 = 0$ . Therefore, we must throw out the 3.

So the solution to the equation is simply {2}.

- c. Lastly, we have a more challenging problem. Here the LCD is quite large, however, we still work the process the same way as always. The LCD here is  $(y - 1)(y - 2)(y - 3)$ . We proceed as follows.

$$\frac{5}{y - 1} + \frac{3}{y - 3} = \frac{8}{y - 2} \quad \text{Multiply by the LCD}$$

$$(y - 1)(y - 2)(y - 3) \cdot \left( \frac{5}{y - 1} + \frac{3}{y - 3} \right) = \left( \frac{8}{y - 2} \right) \cdot (y - 1)(y - 2)(y - 3) \quad \text{Distribute the LCD}$$

$$\frac{5(y - 1)(y - 2)(y - 3)}{y - 1} + \frac{3(y - 1)(y - 2)(y - 3)}{y - 3} = \frac{8(y - 1)(y - 2)(y - 3)}{y - 2} \quad \text{Cancel common factors}$$

$$5(y - 2)(y - 3) + 3(y - 1)(y - 2) = 8(y - 1)(y - 3) \quad \text{Multiply the binomials}$$

$$5(y^2 - 5y + 6) + 3(y^2 - 3y + 2) = 8(y^2 - 4y + 3) \quad \text{Distribute the constants}$$

$$5y^2 - 25y + 30 + 3y^2 - 9y + 6 = 8y^2 - 32y + 24 \quad \text{Combine like terms}$$

$$8y^2 - 34y + 36 = 8y^2 - 32y + 24$$

Even though, here, it looks like an equation that will require factoring to solve (since it contains a second power), it turns out if we subtract the  $8y^2$  from both sides, they cancel out, leaving us with just a simple linear equation to solve as usual. We have

$$\begin{array}{r} 8y^2 - 34y + 36 = 8y^2 - 32y + 24 \\ -8y^2 \quad -8y^2 \end{array}$$

$$\begin{array}{r} -34y + 36 = -32y + 24 \\ +32y \quad +32y \end{array} \quad \text{Add 32y to both sides}$$

$$\begin{array}{r} -2y + 36 = 24 \\ -36 \quad -36 \end{array} \quad \text{Subtract 36 from both sides}$$

$$\frac{-2y}{-2} = \frac{-12}{-2}$$

$$y = 6$$

Then we check for zeros in the denominator. Clearly, 6 will not cause any of the original denominators to be zero. Therefore, the solution set is  $\{6\}$ .

## 5.6 Exercises

Solve.

$$1. \frac{3x}{8} + \frac{1}{2} = \frac{x}{4} + 2$$

$$2. \frac{x}{6} + \frac{3}{2} = \frac{2x}{3} - 1$$

$$3. \frac{x}{3} + 2 = \frac{x}{2} - 1$$

$$4. \frac{7x}{5} + \frac{5}{2} = \frac{x}{10} + \frac{1}{5}$$

$$5. \frac{5}{8x} - \frac{1}{2} = \frac{7}{6x}$$

$$6. \frac{3}{4x} - \frac{1}{2} = \frac{2}{3x} - 2$$

$$7. \frac{1}{6} + \frac{1}{y} = \frac{1}{3}$$

$$8. \frac{1}{x} + \frac{2}{3} = 2$$

$$9. x - 7 + \frac{14}{x} = 2$$

$$10. \frac{x}{2} + \frac{12}{x} = 5$$

$$11. \frac{3}{x-7} = \frac{2}{4x+1}$$

$$12. \frac{7}{x+1} = \frac{6}{x-5}$$

$$13. \frac{4}{x^2-5x-36} = \frac{-1}{x+4}$$

$$14. \frac{x-2}{3x+5} = \frac{6}{x-3}$$

$$15. \frac{x}{x+4} = \frac{2}{x}$$

$$16. \frac{x-1}{7x+6} = \frac{2}{x+2}$$

$$17. \frac{3}{2x-9} - 2 = \frac{5x}{2x-9}$$

$$18. \frac{3}{x-3} - 4 = \frac{x}{x-3}$$

$$19. \frac{x}{x-2} = 3 - \frac{2}{x-2}$$

$$20. \frac{3x}{x+9} + 1 = \frac{x}{x+9}$$

$$21. 3 - \frac{2}{4x-1} = \frac{3x}{4x-1}$$

$$22. 5 - \frac{3}{2x-1} = \frac{x-4}{2x-1}$$

$$23. 8 + \frac{6}{x-5} = \frac{x-13}{x-5}$$

$$24. 1 + \frac{x}{2-x} = \frac{6x}{2-x}$$

$$25. x + \frac{x+5}{x-7} = \frac{10x-58}{x-7}$$

$$26. 1 - \frac{13}{x} + \frac{36}{x^2} = 0$$

$$27. \frac{3}{x+1} - \frac{1}{x+4} = \frac{x+12}{x^2+5x+4}$$

$$28. \frac{6}{x+2} + \frac{5}{x-2} = \frac{20}{x^2-4}$$

$$29. \frac{7}{x+5} + \frac{3}{x-5} = \frac{30}{x^2-25}$$

$$30. \frac{3}{x+4} - \frac{1}{x+3} = \frac{x+9}{x^2+7x+12}$$

31. 
$$\frac{3}{z^2-z-2} + \frac{18}{z^2-2z-3} = \frac{z+21}{z^2-z-2}$$

33. 
$$\frac{3x}{x-4} = \frac{2x}{x-3} + \frac{6}{x^2-7x+12}$$

35. 
$$\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2-6x+8}$$

37. 
$$\frac{x+2}{x+3} - 1 = \frac{1}{3-2x-x^2}$$

39. 
$$\frac{x+2}{x^2-6x-27} + \frac{x-17}{x^2-2x-63} = \frac{x+1}{x^2+10x+21}$$

41. 
$$\frac{3x-2}{x^2-17x+72} - \frac{x-4}{x^2-20x+96} = \frac{x-6}{x^2-21x+108}$$

43. 
$$\frac{x+3}{x^2-6x-16} + \frac{x-14}{x^2-4x-32} = \frac{x+1}{x^2+6x+8}$$

45. 
$$\frac{x-4}{x-7} + \frac{x-5}{x-6} = \frac{x+164}{x^2-13x+42}$$

47. 
$$\frac{x+5}{x^2-6x-27} + \frac{x-17}{x^2-18x+77} = \frac{x+1}{x^2-14x+33}$$

49. 
$$\frac{3x-4}{x^2-10x+21} - \frac{x-8}{x^2-18x+77} = \frac{x-5}{x^2-14x+33}$$

32. 
$$\frac{2}{y^2-y-2} + \frac{18}{y^2-2y-3} = \frac{y+20}{y^2-y-2}$$

34. 
$$\frac{4}{x^2-x-2} + \frac{40}{x^2-2x-3} = \frac{x+44}{x^2-x-2}$$

36. 
$$\frac{x-7}{x-4} + \frac{x+8}{x-3} = \frac{x+158}{x^2-7x+12}$$

38. 
$$\frac{x-5}{x-8} + \frac{x-6}{x-7} = \frac{x+97}{x^2-15x+56}$$

40. 
$$\frac{x+3}{x^2-6x-16} + \frac{x-14}{x^2-4x-32} = \frac{x+1}{x^2+6x+8}$$

42. 
$$\frac{3x-3}{x^2-13x+40} - \frac{x-6}{x^2-22x+112} = \frac{x-9}{x^2-19x+70}$$

44. 
$$\frac{x-6}{x-4} + \frac{x-7}{x-3} = \frac{x+111}{x^2-7x+12}$$

46. 
$$\frac{x-4}{x-8} + \frac{x-5}{x-7} = \frac{x+143}{x^2-15x+56}$$

48. 
$$\frac{x+7}{x^2-3x-18} + \frac{x-17}{x^2-x-30} = \frac{x+1}{x^2+8x+15}$$

50. 
$$\frac{3x-3}{x^2-13x+40} - \frac{x-7}{x^2-19x+70} = \frac{x-2}{x^2-22x+112}$$