

5.5 Complex Fractions

At this point, right after we cover all the basic operations, we would usually turn our attention to solving equations. However, there is one other type of rational expression that we need to address. That is, the complex fraction.

Definition: Complex Fraction- a fraction containing fractions.

For example,

$$\frac{\frac{1}{2}}{\frac{1}{4}}, \quad \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{z}}, \quad \frac{2 - \frac{11}{2x-1}}{3 - \frac{17}{2x-1}}$$

Although this seems like it would be very complicated, simplifying complex fractions is actually quite simple. A complex fraction is simplified when it no longer contains fractions in the numerator and denominator, and when it is fully reduced.

We have a three step process for simplifying complex fractions. It is given below.

Simplifying Complex Fractions

1. Find the LCD of all the fractions in the expression.
2. Multiply numerator and denominator by the LCD.
3. Reduce to lowest terms.

The trickiest part, and most critical part, of this process is the step where we multiply numerator and denominator by the LCD (step 2). The thing that makes it difficult is that you have to multiply the entire “big” numerator and “big” denominator, term by term, by the LCD.

The best way to remember and to write out this step properly is by remembering the phrase “**Top-Top: Bottom-Top**”. It sounds a little funny, however, it does a very good job of describing the “look” of this critical second step.

The idea is, you have to multiply the numerators, numerators by the LCD (that is the Top’s Top’s) and the denominators, numerators by the LCD (that is the Bottom’s Tops).

So, if you can understand what the phrase “Top-Top: Bottom-Top” means, you can make it through the key step in the simplifying of Complex Fraction.

Let’s look at some examples to clarify all of this.

Example 1:

Simplify.

$$\text{a. } \frac{\frac{1}{2}}{\frac{1}{4}}, \quad \text{b. } \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{z}}$$

Solution:

- a. To simplify the complex fraction we will follow the steps as outlined above. We start by finding the LCD of all of the fractions in the expression. In this case, the LCD is clearly 4.

Now, we multiply numerator and denominator by the LCD of 4 as follows

$$\frac{\frac{1}{2} \cdot 4}{\frac{1}{4} \cdot 4}$$

So, to multiply, we remember that 4 is the same as $\frac{4}{1}$. We, therefore, multiply straight across as we do with standard fractions. This gives us,

$$\frac{\frac{1}{2} \cdot \frac{4}{1}}{\frac{1}{4} \cdot \frac{4}{1}}$$

$$= \frac{\frac{4}{2}}{\frac{4}{4}}$$

This is the reason we refer to this step as “Top-Top: Bottom-Top”. It perfectly describes what the step looks like. It looks like we simply multiplied the “Top’s Top’s” (the numerators, numerators) and the “Bottom’s Top’s” (the denominators, denominators) by the LCD. Now we simplify by reducing as much as possible

$$\frac{\frac{4}{2}}{\frac{4}{4}}$$

$$= \frac{2}{1}$$

$$= 2$$

- b. This time, the problem is a little more challenging; however, the technique remains the same. We start by noticing that the LCD is xyz . Then, multiply the numerator and denominator by xyz , that is “Top-Top: Bottom-Top”.

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{z}}$$

$$= \frac{\left(\frac{1}{x} + \frac{1}{y}\right)xyz}{\left(\frac{1}{z}\right)xyz}$$

Multiply by distributing xyz to the numerators

$$= \frac{\frac{xyz}{x} + \frac{xyz}{y}}{\frac{xyz}{z}}$$

Top-Top: Bottom-Top

Now, notice we have some canceling. Always remember, you can ONLY cancel when we have multiplication. That means, we can cancel on the, what we will call “pockets” of multiplication, meaning $\frac{xyz}{x}$ is all multiplication so we can cancel the x ’s, for example. Once we do this initial canceling, we can try to reduce by factoring.

$$\frac{\frac{xyz}{x} + \frac{xyz}{y}}{\frac{xyz}{z}}$$

$$= \frac{yz + xz}{xy}$$

$$= \frac{z(y + x)}{xy}$$

Since nothing cancels, this is our answer.

Now let's look at some more challenging complex fractions in the next example.

Example 2:

Simplify.

a. $\frac{2 - \frac{11}{2x-1}}{3 - \frac{17}{2x-1}}$

b. $\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$

c. $\frac{\frac{7}{x-4} - \frac{5}{x+3}}{\frac{x-4}{5} + \frac{x+3}{x-4}}$

d. $\frac{x+3 - \frac{18}{2x+1}}{x - \frac{6}{2x+1}}$

Solution:

- a. As we did in example 1 above, we start by finding the LCD and then multiply the LCD by the numerator and denominator, Top-Top: Bottom-Top. Here the LCD is $2x - 1$.

$$\begin{aligned} & \frac{2 - \frac{11}{2x-1}}{3 - \frac{17}{2x-1}} \\ &= \frac{\left(2 - \frac{11}{2x-1}\right) \cdot (2x-1)}{\left(3 - \frac{17}{2x-1}\right) \cdot (2x-1)} && \text{Distribute the LCD to the Top-Top and Bottom-Top} \\ &= \frac{2(2x-1) - \frac{11(2x-1)}{\cancel{2x-1}}}{3(2x-1) - \frac{17(2x-1)}{\cancel{2x-1}}} && \text{Multiply out the front terms and cancel the } 2x-1 \text{ in our multiplication "pockets"} \\ &= \frac{4x - 2 - 11}{6x - 3 - 17} && \text{Combine like terms} \\ &= \frac{4x - 13}{6x - 20} \\ &= \frac{4x - 13}{2(3x - 10)} && \text{Factor to see if we can reduce} \end{aligned}$$

Notice in the third step above, we didn't cancel the $(2x - 1)$ in the front of the expression. The reason is, we did not have ONLY multiplication. The term $2(2x - 1)$ and the term $3(2x - 1)$ are subtracted from the rest of the numerator and denominator. Therefore, they are multiplied.

In general, it's always best to cancel in the "pockets" of multiplication, and immediately multiply out what doesn't cancel. This way we avoid canceling when we can't.

- b. This time our LCD is a^2b^2 . So, we start by multiplying the LCD by the numerator and denominator, then simplify.

$$\begin{aligned}
 & \frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} \\
 &= \frac{\left(\frac{1}{a} - \frac{1}{b}\right) a^2 b^2}{\left(\frac{1}{a^2} - \frac{1}{b^2}\right) a^2 b^2} \quad \text{Distribute the LCD to the Top-Top and Bottom-Top} \\
 &= \frac{\frac{a^2 b^2}{a} - \frac{a^2 b^2}{b}}{\frac{a^2 b^2}{a^2} - \frac{a^2 b^2}{b^2}} \quad \text{Cancel what we can in the multiplication "pockets"} \\
 &= \frac{ab^2 - a^2 b}{b^2 - a^2} \\
 &= \frac{\cancel{ab}(b - \cancel{a})}{(b - \cancel{a})(b + a)} \quad \text{Factor to see if we can reduce} \\
 &= \frac{ab}{b + a} \quad \text{Cancel the } b - a \text{ to reduce}
 \end{aligned}$$

- c. Again, we follow the same technique as the other examples. This time the LCD is $(x - 4)(x + 3)$. The steps are as follows.

$$\begin{aligned}
 & \frac{\frac{7}{x-4} - \frac{5}{x+3}}{\frac{5}{x+3} + \frac{9}{x-4}} \\
 &= \frac{\left(\frac{7}{x-4} - \frac{5}{x+3}\right)(x-4)(x+3)}{\left(\frac{5}{x+3} + \frac{9}{x-4}\right)(x-4)(x+3)} \quad \text{Distribute the LCD to the Top-Top and Bottom-Top} \\
 &= \frac{\frac{7(x-4)(x+3)}{x-4} - \frac{5(x-4)(x+3)}{x+3}}{\frac{5(x-4)(x+3)}{x+3} + \frac{9(x-4)(x+3)}{x-4}} \quad \text{Cancel what we can in the multiplication "pockets"} \\
 &= \frac{7(x+3) - 5(x-4)}{5(x-4) + 9(x+3)} \quad \text{Careful here, you can't cancel, instead, multiply everything out.} \\
 &= \frac{7x + 21 - 5x + 20}{5x - 20 + 9x + 27} \\
 &= \frac{2x + 41}{14x + 7} \quad \text{Factor to see if we can reduce} \\
 &= \frac{2x + 41}{7(2x + 1)}
 \end{aligned}$$

Since nothing cancels, this is our answer.

- d. Lastly, our LCD is simply $2x + 1$. So, just like all of the others, we will multiply by the LCD, and simplify down as far as we can go. We proceed as follows

$$\frac{x + 3 - \frac{18}{2x+1}}{x - \frac{6}{2x+1}}$$

$$= \frac{\left(x+3 - \frac{18}{2x+1}\right)}{\left(x - \frac{6}{2x+1}\right)} \quad \frac{2x+1}{2x+1}$$

Distribute the LCD to all the Top-Top's and Bottom-Top's

$$= \frac{x(2x+1) + 3(2x+1) - \frac{18(2x+1)}{2x+1}}{x(2x+1) - \frac{6(2x+1)}{2x+1}}$$

Cancel in the multiplication "pockets" and multiply out everything else

$$= \frac{2x^2 + x + 6x + 3 - 18}{2x^2 + x - 6}$$

Combine like terms

$$= \frac{2x^2 + 7x - 15}{2x^2 + x - 6}$$

$$= \frac{\cancel{(2x-3)}(x+5)}{\cancel{(2x-3)}(x+2)}$$

Factor and reduce

$$= \frac{x+5}{x+2}$$

5.5 Exercises

Simplify.

$$1. \frac{\frac{3}{4} + \frac{2}{3}}{\frac{5}{2} - \frac{1}{3}}$$

$$2. \frac{\frac{1}{5} + \frac{1}{3}}{\frac{5}{6} - \frac{1}{3}}$$

$$3. \frac{3 - \frac{6}{7}}{\frac{4}{3} + \frac{1}{4}}$$

$$4. \frac{5 - \frac{7}{8}}{\frac{6}{5} + \frac{1}{6}}$$

$$5. \frac{6 + \frac{3}{z}}{z + \frac{1}{2}}$$

$$6. \frac{x + \frac{2}{7}}{x + \frac{3}{11}}$$

$$7. \frac{\frac{5}{x+4} - 2}{\frac{1}{x+4} + 3}$$

$$8. \frac{\frac{6}{x-5} + 7}{\frac{8}{x-5} - 9}$$

$$9. \frac{5 - \frac{3}{2x-1}}{\frac{2}{2x-1} + 3}$$

$$10. \frac{10 - \frac{5}{x}}{\frac{1}{x} + 25}$$

$$11. \frac{4 + \frac{8}{x}}{1 - \frac{4}{x^2}}$$

$$12. \frac{2 + \frac{10}{x}}{1 - \frac{25}{x^2}}$$

$$13. \frac{1 + \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{5}{x} - \frac{36}{x^2}}$$

$$14. \frac{1 + \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{3}{x} - \frac{28}{x^2}}$$

$$15. \frac{6 - \frac{36}{x} + \frac{54}{x^2}}{\frac{9}{x^2} - 1}$$

$$16. \frac{\frac{50}{x^2} - \frac{20}{x} + 2}{\frac{25}{x^2} - 1}$$

$$17. \frac{\frac{8}{x+3} + \frac{4}{x+9}}{\frac{x+7}{x+3}}$$

$$18. \frac{\frac{4}{x+7} + \frac{6}{x+2}}{\frac{x+5}{x+7}}$$

$$19. \frac{\frac{15}{x} + \frac{3}{x-6}}{\frac{x}{x+5} + \frac{x-6}{2}}$$

$$20. \frac{\frac{42}{x} + \frac{6}{x-8}}{\frac{x}{x+7} + \frac{x-8}{2}}$$

$$21. \frac{\frac{12}{x} + \frac{4}{x-4}}{\frac{x}{x-4} + \frac{x}{x+3}}$$

$$22. \frac{\frac{10}{x} + \frac{5}{x-3}}{\frac{x}{x+2} + \frac{x-3}{2}}$$

$$23. \frac{\frac{3}{x-2} + \frac{4}{x-4}}{\frac{5}{x-4} - \frac{x-2}{2}}$$

$$24. \frac{\frac{4}{x-7} + \frac{4}{x-2}}{\frac{x-7}{5} - \frac{x-2}{1}}$$

$$25. \frac{\frac{1}{2x-1} - \frac{2}{x+4}}{\frac{2}{2x-1} - \frac{1}{x+4}}$$

$$26. \frac{\frac{7}{x-2} - \frac{2}{x+2}}{\frac{5}{x+2} - \frac{1}{x-2}}$$

$$27. \frac{\frac{x}{x+5} + \frac{3}{x+2}}{\frac{2}{x+2} - \frac{x}{x+5}}$$

$$28. \frac{\frac{2x}{x-4} - \frac{1}{x+2}}{\frac{2}{x+2} - \frac{x}{x-4}}$$

$$29. \frac{\frac{5}{x+2} - \frac{3}{x-2}}{\frac{x}{x-1} + \frac{x}{x+1}}$$

$$30. \frac{\frac{x}{x+5} + \frac{3x}{2x}}{\frac{2x}{x} - \frac{x}{x+5}}$$

$$31. \frac{x-8 + \frac{20}{x+4}}{x-10 + \frac{24}{x+4}}$$

$$32. \frac{x-3 + \frac{2}{x}}{x-4 + \frac{3}{x}}$$

$$33. \frac{x+3 - \frac{20}{x-5}}{x+8 + \frac{30}{x-5}}$$

$$34. \frac{x+2 + \frac{4}{x-3}}{x-1 + \frac{2}{x-3}}$$

$$35. \frac{\frac{y^2-y-2}{y^2-3y-4}}{\frac{y^2+4y+3}{y^2-y-12}}$$

$$36. \frac{\frac{64x^2-64x-9}{7x-x^2}}{\frac{64x^2-1}{x^2-4x-21}}$$

$$37. \frac{\frac{1}{x^2+3x+2} + \frac{1}{3x}}{\frac{x^2+3x+2}{3x} - \frac{1}{x+2}}$$

$$38. \frac{\frac{1}{x^2+3x} + \frac{1}{x^2+x}}{\frac{x^2+3x}{x} - \frac{1}{x+3}}$$

$$39. \frac{\frac{1}{x^2-1} - \frac{2}{x^2+4x+3}}{\frac{x^2-1}{x^2+2x-3} + \frac{1}{x+3}}$$

$$40. \frac{\frac{1}{x^2+4x+4} - \frac{1}{x^2-4x+4}}{\frac{2}{x^2-4} - \frac{1}{x+2}}$$

$$41. \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$42. \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$43. \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$$

$$44. \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}$$

$$45. \frac{\frac{1}{(x+h-1)^2} - \frac{1}{(x-1)^2}}{h}$$