

5.4 Adding and Subtracting Rational Expressions with unlike denominators

Now that we can add or subtract rational expressions which have the same denominators and we can find the LCD, we are ready to add or subtract rational expressions with different denominators.

Again, the idea is very similar to that of standard fractions. What we will need to do is make each rational expression have the LCD by multiplying the numerator and denominator by whatever it takes to make each expression have the LCD.

Recall that in 5.1 we had a property (called the Fundamental Principle of Rational Expressions) that will actually allow us to do this.

Here are the steps we will use to do the adding and subtracting.

General Addition and Subtraction of Rational Expressions

1. Find the LCD.
2. Write each rational expression by whatever it takes so that it will be the LCD by using the Fundamental Principle of Rational Expressions, namely $\frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$.
3. Add or subtract numerators over the common denominator.
4. Reduce to lowest terms.

Be forewarned, adding and subtracting rational expressions can be very lengthy and can end up with messy answers. Always keep in mind, "an ugly answer is still an answer".

Let's look at some examples. We will start with the slightly more unusual cases in the first example, and then the examples that follow (examples 2 and 3) will be the more "standard" types of cases.

Example 1:

Simplify.

a. $\frac{7x}{x-6} + \frac{42}{6-x}$

b. $\frac{14}{15-5x} - \frac{8}{2x-6}$

Solution:

- a. First, we need to notice that the denominators are almost the same. In fact, they are clearly only off by a negative. So in this case, we will need to factor the negative out of one of the denominators to make them the same. Let's factor the negative out of the second one. We get

$$\begin{aligned} \frac{7x}{x-6} + \frac{42}{6-x} \\ = \frac{7x}{x-6} + \frac{42}{-(x-6)} \end{aligned}$$

At this point, we need to recall a property from chapter 1. It was one of the division properties which said, $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$. This means that if a fraction (or rational expression) has a negative, we can place the negative on top, bottom or out front. In our current situation, it would be best to move the negative out front and combine it with the + that is already there. Making our problem now look like

$$\begin{aligned} \frac{7x}{x-6} + \frac{42}{-(x-6)} \\ = \frac{7x}{x-6} - \frac{42}{x-6} \end{aligned}$$

Now we can subtract as we did in the last section.

$$\begin{aligned} \frac{7x}{x-6} - \frac{42}{x-6} \\ = \frac{7x-42}{x-6} \\ = \frac{7(x-6)}{x-6} \\ = 7 \end{aligned}$$

- b. This time, we will need to start by finding the LCD. Recall, from the last section, this requires us to factor each denominator. This gives

$$\begin{aligned} \frac{14}{15-5x} - \frac{8}{2x-6} \\ = \frac{14}{5(3-x)} - \frac{8}{2(x-3)} \end{aligned}$$

As we saw in part a, the denominators have a similarity. We will need to factor the negative out of one of them to make the LCD easier to find. Lets again factor the negative out of the back numerator, and we will again combine it with the subtraction out in front of the fraction. We get

$$\begin{aligned} \frac{14}{5(3-x)} - \frac{8}{2(x-3)} \\ = \frac{14}{5(3-x)} - \frac{8}{-2(3-x)} \\ = \frac{14}{5(3-x)} + \frac{8}{2(3-x)} \end{aligned}$$

Now we can clearly see that our LCD is $10(3-x)$. So, we will need to multiply numerator and denominator of each by whatever it takes to make each have that LCD. Namely, multiply the first fraction by 2 and the last one by 5. We have

$$\begin{aligned} \frac{14 \cdot 2}{5(3-x) \cdot 2} + \frac{8 \cdot 5}{2(3-x) \cdot 5} \\ = \frac{28}{10(3-x)} + \frac{40}{10(3-x)} \end{aligned}$$

Finally, we add the numerators and reduce.

$$\begin{aligned} \frac{28}{10(3-x)} + \frac{40}{10(3-x)} \\ = \frac{68}{10(3-x)} \\ = \frac{34}{5(3-x)} \end{aligned}$$

Now we will take a look at some adding and subtracting that are a little more atypical.

Example 2:

Simplify.

a. $\frac{x}{x+2} - \frac{3x}{4x-1}$

b. $\frac{4z}{z^2+z-20} + \frac{z}{z^2-8z+16}$

Solution:

- a. We will use the process that we laid out in the section. This means we begin by finding the LCD. Since in this case, the denominators are already factored, we have an LCD of $(x + 2)(4x - 1)$.

Now, we have to multiply numerator and denominator of each rational expression by whatever it takes to make each denominator have this LCD. What that means is, we will need to **multiply each top and bottom by what it is missing from the LCD**. So we will multiply the first one by $4x - 1$ and the second one by $x + 2$. We get

$$\begin{aligned} & \frac{x}{x+2} - \frac{3x}{4x-1} \\ &= \frac{x}{(x+2)(4x-1)} - \frac{3x}{(4x-1)(x+2)} \\ &= \frac{4x^2 - x}{(x+2)(4x-1)} - \frac{3x^2 + 6x}{(4x-1)(x+2)} \end{aligned}$$

Now, the denominators are the same, so as we did in the last section, we simply need to (carefully) subtract the numerators over the LCD and then see if we can reduce.

$$\begin{aligned} & \frac{4x^2 - x}{(x+2)(4x-1)} - \frac{3x^2 + 6x}{(4x-1)(x+2)} \\ &= \frac{4x^2 - x - (3x^2 + 6x)}{(x+2)(4x-1)} \\ &= \frac{4x^2 - x - 3x^2 - 6x}{(x+2)(4x-1)} \\ &= \frac{4x^2 - 7x}{(x+2)(4x-1)} \\ &= \frac{x(4x-7)}{(x+2)(4x-1)} \end{aligned}$$

Since nothing cancels, this is our answer.

- b. Again, the first thing we need to do is find the LCD. This time, we will need to factor all the numerators first to determine what the LCD is.

$$\begin{aligned} & \frac{4z}{z^2+z-20} + \frac{z}{z^2-8z+16} \\ &= \frac{4z}{(z-4)(z+5)} + \frac{z}{(z-4)^2} \end{aligned}$$

So the LCD is $(z + 5)(z - 4)^2$.

Now, as we did in part a, we multiply top and bottom of each rational expression by what it is missing from the LCD. In this case, we need to multiply the first expression by one more $z - 4$, and we need to multiply the second one by $z + 5$. This gives

$$\begin{aligned} & \frac{4z}{(z-4)(z+5)} \frac{(z-4)}{(z-4)} + \frac{z}{(z-4)^2(z+5)} \frac{(z+5)}{(z+5)} \\ &= \frac{4z^2 - 16z}{(z-4)^2(z+5)} + \frac{z^2 + 5z}{(z-4)^2(z+5)} \end{aligned}$$

Now add the numerators together and factor to see if there is any reducing.

$$\begin{aligned} & \frac{4z^2 - 16z}{(z-4)^2(z+5)} + \frac{z^2 + 5z}{(z-4)^2(z+5)} \\ &= \frac{4z^2 - 16z + z^2 + 5z}{(z-4)^2(z+5)} \\ &= \frac{5z^2 - 11z}{(z-4)^2(z+5)} \\ &= \frac{z(5z - 11)}{(z-4)^2(z+5)} \end{aligned}$$

Again, since nothing cancels, this is the answer.

In the final example, we will take a look at the more challenging problems, namely, where there are three rational expressions to add and/or subtract. The good news is, it is done the exact same way as above.

Example 3:

Simplify.

$$\text{a. } \frac{4a}{a^2+2a-15} + \frac{3a}{2a^2+11a+5} - \frac{5a}{2a^2-5a-3} \qquad \text{b. } \frac{3x+1}{x-1} - \frac{x-1}{x-3} + \frac{x+1}{x^2-4x+3}$$

Solution:

a. First, we need to factor all the denominators to determine the LCD.

$$\begin{aligned} & \frac{4a}{a^2+2a-15} + \frac{3a}{2a^2+11a+5} - \frac{5a}{2a^2-5a-3} \\ &= \frac{4a}{(a+5)(a-3)} + \frac{3a}{(2a+1)(a+5)} - \frac{5a}{(2a+1)(a-3)} \end{aligned}$$

Our LCD is $(a+5)(2a+1)(a-3)$. Now, we make each have the LCD by multiplying each numerator and denominator by what the fraction is missing from the LCD. In this case, the first one needs the $2a+1$, the second needs $a-3$ and the last needs $a+5$. We get

$$\begin{aligned} & \frac{4a}{(a+5)(a-3)} \frac{(2a+1)}{(2a+1)} + \frac{3a}{(2a+1)(a+5)} \frac{(a-3)}{(a-3)} - \frac{5a}{(2a+1)(a-3)} \frac{(a+5)}{(a+5)} \\ &= \frac{8a^2+4a}{(a+5)(a-3)(2a+1)} + \frac{3a^2+9a}{(2a+1)(a+5)(a-3)} - \frac{5a^2+25a}{(2a+1)(a-3)(a+5)} \end{aligned}$$

Now, add and subtract numerators and try to reduce by factoring.

$$\begin{aligned}
&= \frac{8a^2 + 4a + 3a^2 + 9a - (5a^2 + 25a)}{(a + 5)(2a + 1)(a - 3)} \\
&= \frac{8a^2 + 4a + 3a^2 + 9a - 5a^2 - 25a}{(a + 5)(2a + 1)(a - 3)} \\
&= \frac{6a^2 - 12a}{(a + 5)(2a + 1)(a - 3)} \\
&= \frac{6a(a - 2)}{(a + 5)(2a + 1)(a - 3)}
\end{aligned}$$

Nothing reduces so this is our answer.

- b. Finally, we proceed just like all of the other examples. We start by factoring to find our LCD.

$$\begin{aligned}
&\frac{3x + 1}{x - 1} - \frac{x - 1}{x - 3} + \frac{x + 1}{x^2 - 4x + 3} \\
&= \frac{3x + 1}{x - 1} - \frac{x - 1}{x - 3} + \frac{x + 1}{(x - 1)(x - 3)}
\end{aligned}$$

So the LCD is $(x - 1)(x - 3)$. We continue the process like we did for the others. Multiply each by what it is missing, add and subtract numerators and try to reduce.

$$\begin{aligned}
&\frac{3x+1}{x-1} - \frac{x-1}{x-3} + \frac{x+1}{(x-1)(x-3)} \\
&= \frac{(3x+1)(x-3)}{(x-1)(x-3)} - \frac{(x-1)(x-1)}{(x-3)(x-1)} + \frac{x+1}{(x-1)(x-3)} \\
&= \frac{3x^2 - 9x + x - 3}{(x-1)(x-3)} - \frac{x^2 - x - x + 1}{(x-1)(x-3)} + \frac{x+1}{(x-1)(x-3)} \\
&= \frac{3x^2 - 8x - 3}{(x-1)(x-3)} - \frac{x^2 - 2x + 1}{(x-1)(x-3)} + \frac{x+1}{(x-1)(x-3)} \\
&= \frac{3x^2 - 8x - 3 - (x^2 - 2x + 1) + x + 1}{(x-1)(x-3)} \\
&= \frac{3x^2 - 8x - 3 - x^2 + 2x - 1 + x + 1}{(x-1)(x-3)} \\
&= \frac{2x^2 - 5x - 3}{(x-1)(x-3)} \\
&= \frac{(2x+1)(x-3)}{(x-1)(x-3)} \\
&= \frac{2x+1}{x-1}
\end{aligned}$$

Multiply out using techniques from chapter 3

Combining like terms

Add/Subtracting numerators

Combining like terms again

Factor and reduce

5.4 Exercises

Simplify.

1. $\frac{3x}{x-4} + \frac{12}{4-x}$

2. $\frac{5x}{x-2} + \frac{10}{2-x}$

3. $\frac{y^2+8y}{y-1} - \frac{2y-11}{1-y}$

4. $\frac{y^2+6y}{y-3} - \frac{4y-39}{3-y}$

5. $\frac{x^2+x-27}{x-4} + \frac{x^2-8x+9}{4-x}$

6. $\frac{x^2+3x-53}{x-7} + \frac{x^2-5x+3}{7-x}$

7. $\frac{x^2-7x+23}{x^2-36} + \frac{6x-19}{36-x^2}$

8. $\frac{x^2+9x}{x^2-1} - \frac{3x-13}{1-x^2}$

9. $\frac{3x+14}{x^2-8x} - \frac{x^2-15x+18}{8x-x^2}$

10. $\frac{x^2+4x-49}{x^2-7x} + \frac{x^2-4x+7}{7x-x^2}$

11. $\frac{5}{x-6} + \frac{3}{x+5}$

12. $\frac{9}{y+1} - \frac{4}{y-1}$

13. $\frac{3x+12}{x^2+x-12} + \frac{2}{x-3}$

14. $\frac{4x-1}{x^2-2x-8} - \frac{3}{x-4}$

15. $\frac{2x+1}{x^2+2x-8} - \frac{3}{x+4}$

16. $\frac{3x}{x-2} + \frac{6}{x-5}$

17. $\frac{2x}{x-8} + \frac{8}{x-2}$

18. $\frac{2}{x-6} - \frac{5x}{x+5}$

19. $\frac{5}{x-4} + \frac{6x}{x+2}$

20. $\frac{3a}{a+9} - \frac{8a}{a+5}$

21. $\frac{3x}{x+2} - \frac{5}{x-3}$

22. $\frac{5}{x^2-9x-22} + \frac{8}{x^2+4x+4}$

23. $\frac{x}{x^2-9} - \frac{4}{x^2-6x+9}$

24. $\frac{x-2}{x^2-x-56} + \frac{9}{x^2-2x-63}$

25. $\frac{x-4}{x^2-4} + \frac{1}{x^2-2x}$

26. $\frac{3}{x^2+12x+20} + \frac{x-2}{x^2+2x-80}$

27. $\frac{x+4}{x^2+15x+56} + \frac{6}{x^2+16x+63}$

28. $\frac{x+3}{x^2+13x+42} - \frac{8}{x^2+12x+35}$

29. $\frac{x+7y}{2x^2-7xy-15y^2} - \frac{x-4y}{2x^2+19xy+24y^2}$

30. $\frac{x+y}{x^2+5xy-14y^2} - \frac{x+5y}{x^2+17xy+70y^2}$

31. $\frac{x+6}{x^2+6x-27} + \frac{x+10}{x^2+14x+45}$

32. $\frac{x+3}{x^2+16x+63} + \frac{x+6}{x^2+19x+90}$

33. $\frac{x+13}{2x^2-11x-40} - \frac{x-4}{2x^2+23x+45}$

34. $\frac{x}{x^2-1} - \frac{x+1}{x^2-2x+1}$

35. $\frac{x+6}{x^2+5x+6} + \frac{x+9}{x^2+8x+15}$

36. $\frac{x-8}{x^2-4} - \frac{x+2}{x^2-2x}$

37. $\frac{4x}{x+3} - 2$

38. $\frac{y}{y-1} + 1$

39. $1 - \frac{5x+10}{x^2+4x+4}$

40. $\frac{x-4}{2x+1} - 3$

41. $\frac{2x}{x-1} - 1 - \frac{4}{x+1}$

42. $\frac{2x-1}{x+2} + 2 - \frac{3}{x-2}$

$$43. \frac{5x}{x+3} + \frac{2}{x^2-9} - \frac{7x}{x-3}$$

$$45. \frac{2x+1}{x-2} - \frac{x}{x-1} + \frac{x+2}{x^2-3x+2}$$

$$47. \frac{2}{x-3y} + \frac{x-y}{x-4y} - \frac{1}{x^2-7xy+12y^2}$$

$$49. \frac{2x-3}{6x^2+x-2} - \frac{x+1}{2x^2+3x-2} + \frac{x}{3x^2+8x+4}$$

$$44. \frac{x}{x^2-49} + \frac{2x}{x-7} - \frac{7x}{x+7}$$

$$46. \frac{x+2}{x-6} - \frac{2}{x^2-12x+36} - \frac{x-2}{x-6}$$

$$48. \frac{x-1}{x^2+3x} - \frac{2-x}{x^2-9} - \frac{7}{x-3}$$

$$50. \frac{x+3}{2x^2+3x-2} - \frac{2x+1}{2x^2-17x+8} - \frac{x+2}{x^2-6x-16}$$