5.3 Adding and Subtracting Rational Expressions

Now we want to turn our attention to the adding and subtracting of rational expressions. We will do this in two parts, covering the next two sections.

The good news is, as was the case for multiplying and dividing rational expressions, adding and subtracting rational expressions is just like adding and subtracting standard fractions.

Recall that in order to add or subtract fractions, you had to have the same denominators. Once we did, we simply added or subtracted the numerators over the common denominator.

Extending this idea to rational expressions gives us the following property

---

Addition and Subtraction Property

$\frac{P}{R} + \frac{Q}{R} = \frac{P+Q}{R}$ and $\frac{P}{R} - \frac{Q}{R} = \frac{P-Q}{R}$

So the concept behind this is simple. We just have to be very careful in the execution of the concept because we are dealing with multi-termed numerators and denominators. Usually, the problems arise when we are doing subtraction, as we will see. Basically, we must make sure that we are very careful with our negatives.

Also, we have to make sure we always reduce our answers.

Example 1:

Simplify.

    a. $\frac{8}{y^2} + \frac{10}{y^2}$
    b. $\frac{x^2}{x+3} - \frac{x^2+2x-3}{x+3}$

    c. $\frac{x-9}{x^2-4x-12} + \frac{3}{x^2-4x-12}$
    d. $\frac{4a-1}{a^2+7a+6} - \frac{3a-2}{a^2+7a+6}$

Solution:
    a. Since the denominators are the same, we simply need to add the numerators together over the common denominator and see if we can reduce.

$$\frac{8}{y^2} + \frac{10}{y^2}$$

$$= \frac{18}{y^2}$$

    b. Again, the denominators are the same, so we need only to subtract the numerators. However, notice that the second numerator has more than one term. This means we need to make sure we subtract the ENTIRE numerator. That is to say, we should put parenthesis around the second numerator, when we subtract, so that we assure that we have actually subtracted the entire numerator. It looks as follows

$$\frac{x^2}{x+3} - \frac{x^2+2x-3}{x+3}$$

$$= \frac{x^2 - (x^2+2x-3)}{x+3}$$

Now, clearly, we need to distribute the negative and combine like terms.

$$= \frac{x^2 - x^2 - 2x + 3}{x+3}$$

$$= \frac{-2x+3}{x+3}$$

Since nothing factors, the expression is reduced and therefore, is our answer.

c. Once again, the rational expressions already have the same denominator, and so we simply add numerators over the common denominator.

$$\frac{x-9}{x^2-4x-12} + \frac{3}{x^2-4x-12}$$

$$= \frac{x-9+3}{x^2-4x-12}$$

$$= \frac{x-6}{x^2-4x-12}$$

At this point, we need to factor and reduce (by canceling as we did in section 5.1). We get

$$= \frac{\cancel{x-6}}{\cancel{(x-6)}(x+2)}$$

$$= \frac{1}{x+2}$$

d. Lastly, here we notice, yet again, that the denominators are the same, and also, we have subtraction. So, we need to subtract the numerators, carefully, over the common denominator. We have

$$\frac{4a-1}{a^2+7a+6} - \frac{3a-2}{a^2+7a+6}$$

$$= \frac{4a-1-(3a-2)}{a^2+7a+6}$$

$$= \frac{4a-1-3a+2}{a^2+7a+6}$$

$$= \frac{a+1}{a^2+7a+6} \qquad \text{Factor the denominator to try and reduce}$$

$$= \frac{\cancel{a+1}}{\cancel{(a+1)}(a+6)}$$

$$= \frac{1}{a+6}$$

As we saw in example 1, when the denominators are the same, adding and subtracting is not very difficult. However, as you know, usually the denominators do not start out the same.

So, what do we do if they don't have the same denominators?

Just like we did with standard fractions, we need to use an LCD to add and subtract. Therefore, the first thing we need to do is be able to find the LCD when we have variables involved. The following process will help us with this.

Finding the LCD
1. Factor each denominator completely.
2. Choose the **greatest** power of each <u>different</u> factor.
3. The product of these is the LCD.

This process seems somewhat simple, but it may look odd that we are to choose the "greatest" power of each factor. The simple answer is, remember that the LCD is the LCM of the denominators. That means, the LCD needs to be a multiple of all of the denominators.

So, that being said, the LCD must contain all of the denominators. By choosing the "greatest" power, we are assured that it will contain enough for ALL of the denominators.

Let's try finding the LCD with the following example.

<u>Example 2</u>:

Find the LCD of the following denominators.

   a. $6x^2y^2 - 12xy^2$; $9xy - 18y$ b. $z^2 - 2z + 1$; $z^2 - 1$

   c. $x^2 - 2x + 1$; $x^2 - 1$; $x^2 + 2x + 1$ d. $2x^2 - x - 6$; $x^2 + 8x + 16$; $2x^2 + 11x + 12$

Solution:
   a. To find the LCD, we just need to follow the steps given above. So, we begin with factoring all of the denominators.

$$6x^2y^2 - 12xy^2 \qquad\qquad 9xy - 18y$$
$$= 6xy^2(x - 2) \qquad\qquad = 9y(x - 2)$$

To find the LCD, first we will find the LCD of the "number part" of the GCF's. In this case, the LCD of 6 and 9 is 18.

Then, we need to take each different factor. The different factors that we see are x, y, and x - 2. We have to take each of these to the highest power that <u>each</u> has available.

So since x and x - 2 only have the first power, they will only be the first power in the LCD. However, the y has a highest power of 2, so y will need to be squared in the LCD.

So, the LCD is $18xy^2(x - 2)$.

   b. Again, we will need to start by factoring each denominator.

$$z^2 - 2z + 1 \qquad\qquad z^2 - 1$$
$$= (z - 1)(z - 1) \qquad\qquad = (z - 1)(z + 1)$$
$$= (z - 1)^2$$

Clearly, the only different factors that are z - 1 and z + 1. Since the z - 1 has a denominator which is squared, that would be the highest power available for the z - 1.

The z + 1 only shows up once, and it is to the first power, so that would be its highest power.

Therefore the LCD is $(z + 1)(z - 1)^2$

c. This time we have three different denominators. So, we will have to factor all three to determine the LCD.

$$x^2 - 2x + 1 \qquad\qquad x^2 - 1 \qquad\qquad x^2 + 2x + 1$$
$$= (x - 1)(x - 1) \qquad = (x - 1)(x + 1) \qquad = (x + 1)(x + 1)$$
$$= (x - 1)^2 \qquad\qquad\qquad\qquad\qquad\qquad = (x + 1)^2$$

So, to find the LCD, we take each different factor, to its highest power. Since the only factors we have are x - 1 and x + 1, and each has a squared power available.

So the LCD must be $(x - 1)^2(x + 1)^2$.

d. Lastly, we, again, need to factor all three denominators to find the LCD.

$$2x^2 - x - 6 \qquad\qquad x^2 + 8x + 16 \qquad\qquad 2x^2 + 11x + 12$$
$$= (2x + 3)(x - 2) \qquad = (x + 4)^2 \qquad\qquad = (2x + 3)(x + 4)$$

Now, we choose all the different factors to their highest individual power. The different factors are $2x + 3$ (highest power of 1), $x - 2$ (highest power of 1), and $x + 4$ (highest power of 2).

Therefore, our LCD is $(2x + 3)(x - 2)(x + 4)^2$.

Now that we can find the LCD and add and subtract rational expressions which have the same denominators, in the next section we will combine these ideas together to add and subtract rational expressions which have different denominators.

5.3 Exercises

Simplify.

1. $\dfrac{5}{x} + \dfrac{6}{x}$

2. $\dfrac{2}{y} + \dfrac{3}{y}$

3. $\dfrac{12}{x^2 y} - \dfrac{13}{x^2 y}$

4. $\dfrac{4}{abc} - \dfrac{7}{abc}$

5. $\dfrac{x}{x-2} + \dfrac{5}{x-2}$

6. $\dfrac{x}{x-5} - \dfrac{2}{x-5}$

7. $\dfrac{x}{x^2+4x+3} + \dfrac{3}{x^2+4x+3}$

8. $\dfrac{x^2-8x+12}{x^2-5x+6} + \dfrac{x-2}{x^2-5x+6}$

9. $\dfrac{8x}{4x-7} - \dfrac{14}{4x-7}$

10. $\dfrac{a}{a^2-6a+8} - \dfrac{4}{a^2-6a+8}$

11. $\dfrac{3b-15}{b^2-64} - \dfrac{2b-7}{b^2-64}$

12. $\dfrac{x^2-8x}{x^2+x-2} + \dfrac{8x-1}{x^2+x-2}$

13. $\dfrac{3}{x-3} - \dfrac{x}{x-3}$

14. $\dfrac{12}{x-6} - \dfrac{2x}{x-6}$

15. $\dfrac{x^2-4x-72}{x^2+x-20} + \dfrac{x^2-2x-8}{x^2+x-20}$

16. $\dfrac{x^2-4x-1}{x^2-2x-8} + \dfrac{4x-15}{x^2-2x-8}$

17. $\dfrac{x^2-2x-35}{x^2-14x-72} - \dfrac{4x+5}{x^2-14x-72}$

18. $\dfrac{x^2+5x-11}{x^2-4x-21} + \dfrac{2x+23}{x^2-4x-21}$

19. $\dfrac{4xy-16x}{y^2-81} - \dfrac{3xy-7x}{y^2-81}$

20. $\dfrac{5z}{3+z} + \dfrac{7z^2-4z}{3+z}$

21. $\dfrac{x-y}{x^2+5xy+6y^2} + \dfrac{3y}{x^2+5xy+6y^2}$

22. $\dfrac{x^2+5xy-15y^2}{x^2-6xy-27y^2} + \dfrac{2x+27y}{x^2-6xy-27y^2}$

23. $\dfrac{x^2-2xy-50y^2}{6x^2+25xy-66y^2} - \dfrac{2xy+10y^2}{6x^2+25xy-66y^2}$

24. $\dfrac{x^2-2xy-17y^2}{x^2-14xy-51y^2} - \dfrac{2xy+4y^2}{x^2-14xy-51y^2}$

25. $\dfrac{(x-4)(x+2)}{x^2-5x-14} - \dfrac{4(x-4)}{x^2-5x-14}$

26. $\dfrac{(x-4)(x+2)}{x^2-5x-36} - \dfrac{4(x-4)}{x^2-5x-36}$

27. $\dfrac{x^2-2x-100}{x^2+x-20} + \dfrac{x^2-4x}{x^2+x-20} - \dfrac{8}{x^2+x-20}$

28. $\dfrac{x^2+3x-17}{x^2-3x-70} + \dfrac{7x+8}{x^2-3x-70} - \dfrac{4}{x^2-3x-70}$

29. $\dfrac{x^2+3x-30}{6x^2+23x-78} - \dfrac{2x-10}{6x^2+23x-78} - \dfrac{x^2+x-20}{6x^2+23x-78}$

30. $\dfrac{x^2-3x-1}{x^2+x-20} - \dfrac{3x-18}{x^2+x-20} - \dfrac{4x-3}{x^2+x-20}$

Find the LCD of the following denominators.

31. $5x, \ 2x$

32. $6y, \ 8y$

33. $3x^2y, \ 9y^2$

34. $5a^3b^2, \ 10ab^3$

35. $4x+20, \ 3x+6$

36. $3x+15, \ 2x+18$

37. $a^2-9, \ a^2+4a-21$

38. $x^2-16, \ x^2+3x-28$

39. $x^2+2x-8, \ x+4$

40. $x^2+x-12, \ x-3$

41. $x^2+6x+9, \ x^2+8x+15$

42. $x^2+4x+4, \ x^2-x-6$

43. $x^2-16, \ x^2+4x-32, \ x^2-8x+16$

44. $x^2-4, \ x^2+4x+4, \ x^2-2x-8$

45. $a^2+14ab+49b^2, \ a^2+15ab+56b^2, \ a^2+16ab+64b^2$

46. $x^2-xy-2y^2, \ x^2+xy-2y^2, \ x^2-y^2$

47. $x^2-3xy, \ 2x-8y, \ x^2-7xy+12y^2$

48. $x^4+6x^3+9x^2, \ x^3+x^2-6x, \ x^2+4x+3$

49. $6xy^2-4y, \ x^3y^2-2x^2y+x, \ 3x^4y^2-5x^3y+2x^2$

50. $2x^2y-xy-3y, \ x^3-x^2-2x, \ x^2-2x+1$