5.2 Multiplying and Dividing Rational Expressions

As you will see, the operations for rational expressions are not much different than the operations for standard fractions.

In fact, the "idea" of the process, is exactly the same as that of regular fractions.

Let's start with Multiplying.

First, recall the property for multiplying standard fractions.

Multiplication Property
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}$

So, as it turns out, the only difference between multiplying simple fractions and multiplying rational expressions is that with rational expressions, you are working with polynomials.

Remember that when multiplying fractions, we like to cancel first, then apply the Multiplication Property. Also, recall from section 5.1 that we can ONLY cancel when multiplying.

So, in light of all of this, we will need to turn the polynomial in each numerator and denominator into multiplication… that means we will have to factor all numerators and denominators first. Then we will be in a position where we can cancel.

In general, we will use the following process

Multiplying Rational Expressions
1. Factor all numerators and denominators. 2. Cancel any common factors. 3. Multiply out remaining factors.

The short way to remember this process is: **Factor and Cancel**. It is very similar to the process of reducing a rational expression.

Example 1:

Simplify.

a. $\frac{8y+16}{3-y} \cdot \frac{4y-12}{3y+6}$

b. $\frac{4x^2-4}{3x^2-13x-10} \cdot \frac{x^2-6x+5}{4x+4}$

c. $\frac{6m^2-7m+2}{6m^2+5m+1} \cdot \frac{2m^2+m}{4m^2-1} \cdot \frac{12m^2-5m-3}{12m^2-17m+6}$

Solution:

a. As we did in the last section, and as we will do throughout chapter 5, we will stick with the given process. If we deviate from the process, we will tend to make the problem more complicated and thereby end up with the wrong answer.

   So, for this example, the process is simple: Factor everything and see what we can cancel. We begin by factoring

$$\frac{8y+16}{3-y}\cdot\frac{4y-12}{3y+6}$$

$$=\frac{8(y+2)}{3-y}\cdot\frac{4(y-3)}{3(y+2)}$$

Notice that the denominator in the first rational expression has a factor that is almost exactly like a factor in the second numerator. We have seen this before and we should recall that we can make them the same by factoring a negative out of one of them. Let's factor the negative out of the first one.

This gives us

$$\frac{8(y+2)}{-(y-3)}\cdot\frac{4(y-3)}{3(y+2)}$$

Now we simply cancel common factors and multiply out what is left. Always make sure you cancel a factor from top with a factor from the bottom. Never cancel two from the top, or two from the bottom.

In this case, canceling and multiplying gives us

$$-\frac{32}{3}$$

b. Again, to multiply these rational expressions, we simply need to factor everything and cancel any factors that are the same on the top and bottom.

We get

$$\frac{4x^2-4}{3x^2-13x-10}\cdot\frac{x^2-6x+5}{4x+4}$$

$$=\frac{4(x^2-1)}{(3x+2)(x-5)}\cdot\frac{(x-5)(x-1)}{4(x+1)}$$

$$=\frac{\cancel{4}(x-1)\cancel{(x+1)}}{(3x+2)\cancel{(x-5)}}\cdot\frac{\cancel{(x-5)}(x-1)}{\cancel{4}\cancel{(x+1)}}$$

Canceling 4, x+1, and x-5 from top and bottom

$$=\frac{(x-1)^2}{3x+2}$$

c. Finally, we have to, once again, factor and cancel. This gives

$$\frac{6m^2-7m+2}{6m^2+5m+1}\cdot\frac{2m^2+m}{4m^2-1}\cdot\frac{12m^2-5m-3}{12m^2-17m+6}$$

$$=\frac{\cancel{(3m-2)}\cancel{(2m-1)}}{\cancel{(3m+1)}(2m+1)}\cdot\frac{m\cancel{(2m+1)}}{\cancel{(2m-1)}\cancel{(2m+1)}}\cdot\frac{\cancel{(3m+1)}\cancel{(4m-3)}}{\cancel{(3m-2)}\cancel{(4m-3)}}$$

Canceling 3m - 2, 2m - 1, 3m + 1, 4m - 3, and one 2m + 1 from top and bottom

$$=\frac{m}{2m+1}$$

Now, with division, again it is almost the same as it is with standard fractions. First, recall, to divide fractions, we had to invert (or "flip") the fraction in back, and change the problem to a multiplication problem.

That is to say, we used the following property

Division Property

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{P \cdot S}{Q \cdot R}$$

So, to divide rational expressions, we will do the exact same thing. We will invert the fraction after the division symbol and change the problem to a multiplication problem. At that point, we can simply proceed as we did for multiplication above.

Here is the process

Dividing Rational Expressions

1. "Flip" the fraction behind the division symbol and change it to multiplication.
2. Multiply as usual.

Since multiplication is just Factor and cancel, we can simply remember dividing rational expressions as: **Flip, Factor, Cancel**.

Example 2:

Simplify.

a. $\frac{9-3z}{2z+8} \div (6-2z)$ b. $\frac{4x^2-9}{x^2-9x+18} \div \frac{2x^2-5x-12}{x^2-10x+24}$

c. $\frac{y+3}{y+6} \cdot \frac{y^2+3y+2}{y^2+5y+4} \div \frac{y^2+5y+6}{y^2+10y+24}$

Solution:

a. For division, it is very similar to multiplication. However, we must first start with inverting the fraction that is after the division symbol. Then the entire problem becomes like the multiplication that we did in example 1.

   Notice here that in order to "flip" the back fraction, we will need to put it over 1 first. So we have

$$\frac{9-3z}{2z+8} \div (6-2z)$$

$$= \frac{9-3z}{2z+8} \div \frac{6-2z}{1}$$

$$= \frac{9-3z}{2z+8} \cdot \frac{1}{6-2z}$$

   Factoring and canceling gives

$$= \frac{3\cancel{(3-z)}}{2(z+4)} \cdot \frac{1}{2\cancel{(3-z)}}$$

$$= \frac{3}{4(z+4)}$$

b. Again we just need to **Flip, Factor, Cancel**. We proceed as follows

$$\frac{4x^2-9}{x^2-9x+18} \div \frac{2x^2-5x-12}{x^2-10x+24}$$

$$= \frac{4x^2 - 9}{x^2 - 9x + 18} \cdot \frac{x^2 - 10x + 24}{2x^2 - 5x - 12}$$

$$= \frac{(2x-3)(2x+3)}{(x-6)(x-3)} \cdot \frac{(x-6)(x-4)}{(2x+3)(x-4)}$$

Canceling 2x + 3, x - 6, and x - 4 from top and bottom

$$= \frac{2x-3}{x-3}$$

c. Lastly, we begin with Flipping and Factoring. This gives

$$\frac{y+3}{y+6} \cdot \frac{y^2+3y+2}{y^2+5y+4} \div \frac{y^2+5y+6}{y^2+10y+24}$$

$$= \frac{y+3}{y+6} \cdot \frac{y^2+3y+2}{y^2+5y+4} \cdot \frac{y^2+10y+24}{y^2+5y+6}$$

$$= \frac{y+3}{y+6} \cdot \frac{(y+2)(y+1)}{(y+4)(y+1)} \cdot \frac{(y+6)(y+4)}{(y+3)(y+2)}$$

Notice that all the factors cancel out. So, just like in regular fractions, when everything cancels, the answer is 1.

5.2 Exercises

Simplify.

1. $\frac{x^3}{2y^2} \cdot \frac{6y^4}{xy}$

2. $\frac{x^2y}{5x^3y} \cdot \frac{10x^4}{2xy^2}$

3. $\frac{a^4}{18a^2b^2} \cdot \frac{15b^4}{a}$

4. $\frac{a^6b}{9} \cdot \frac{6ab^4}{a^8}$

5. $\frac{11xy^2}{x^2+3x-18} \cdot \frac{x^2-9}{22x^3y^2}$

6. $\frac{x^2-y^2}{x^4y} \cdot \frac{xy^2}{3x+3y}$

7. $\frac{c-1}{4ab} \cdot \frac{6a^2b}{1-c}$

8. $\frac{4-x}{5x} \cdot \frac{x^2+5x}{x^2+x-20}$

9. $\frac{x^2+7x+12}{x-5} \cdot \frac{2x-10}{x+3}$

10. $\frac{x^2+5xy+6y^2}{x+2y} \cdot \frac{5x-3y}{x^2-5xy+6y^2}$

11. $\frac{x^2-5x-14}{x+8} \cdot \frac{x^2-64}{x^2-6x-16}$

12. $\frac{2x^2-13x-15}{2x-3} \cdot \frac{4x-6}{x^2-10x+25}$

13. $\frac{x+4}{x^2-49} \cdot \frac{x^2-10x+21}{x^2+x-12}$

14. $\frac{12x+48}{6x-15} \cdot \frac{4x^2-25}{x^2+9x+20}$

15. $\frac{x-7y}{x^2-25y^2} \cdot \frac{x^2+12xy+35y^2}{x^2-49y^2}$

16. $\frac{8x-40}{40-3x-x^2} \cdot \frac{x-8}{2x^2-8x}$

17. $\frac{36ab^3}{bx^2+bx-12b} \cdot \frac{3x^2-27}{8a^2b}$

18. $\frac{3a^2b-ab^2}{6a} \cdot \frac{9a^3-18a^2}{9a^2-b^2}$

19. $\frac{x^2-x-20}{x^2y^3} \cdot \frac{x^3y^2}{x^2-10x+25}$

20. $\frac{2x^2+5x-7}{x+4} \cdot \frac{x^2 4x}{x^2-2x+1}$

21. $\frac{y}{y^2-1} \cdot \frac{y+1}{y^2-y}$

22. $\frac{6x^2+24x}{2x^2+5x-12} \cdot \frac{4x^2-9}{15x^2}$

23. $\frac{x^2-4x-32}{x^2-8x} \cdot \frac{x^3+2x^2+x}{x^2+5x+4} \cdot \frac{x-1}{x^2-1}$

24. $\frac{9x}{x^2-25} \cdot \frac{x^2+5x}{2x-4} \cdot \frac{x^2+3x-10}{3x^4}$

25. $\frac{xy^2}{4x^2} \div \frac{3y^5}{8x^3y}$

26. $\frac{a^2b}{6b^2} \div \frac{2ab^4}{3a^2b}$

27. $\frac{4n^2m^5}{16nm^2} \div \frac{m^6}{8n^3m}$

28. $\frac{x^3y^3}{x^2y^4} \div \frac{y^9}{x^3y}$

29. $\frac{a^2-b^2}{2a^2b^4} \div \frac{2a+2b}{a^3b}$

30. $\frac{12x^2y^5}{x+5} \div \frac{3x^3y}{x^2-25}$

31. $\frac{x^2-3x-10}{x+7} \div \frac{6x-30}{x+7}$

32. $\frac{4x^2-25}{3x^2-16x+5} \div (10x+25)$

33. $\frac{2x+10}{4-x} \div \frac{x^2-10x+24}{x^2-x-30}$

34. $\frac{12x-36}{9-x^2} \div \frac{8x-24}{x+3}$

35. $\frac{x^2-3x-18}{x^2+8x+16} \div \frac{x^2-9}{x^2+3x-4}$

36. $\frac{a^2}{a^2-7a} \div \frac{1}{a^2-4a-21}$

37. $\frac{x^2+8x+16}{x^2-16} \div \frac{x^2-3x-28}{x-7}$

38. $\frac{2x-16}{x^2+2x-24} \div \frac{x-8}{x+6}$

39. $\frac{x^2-36y^2}{x^2+9xy+14y^2} \div \frac{x^2-15xy+54y^2}{x^2-7xy-18y^2}$

40. $\frac{4x^2+32x+64}{x^2-7x} \div \frac{2x^2-6x-56}{x^2-49}$

41. $\frac{3x^2-9x+6}{3x^2-7x+4} \div \frac{3x^2-12}{3x^2+2x-8}$

42. $\frac{x^2-x-12}{2x^2-15x+18} \div \frac{3x^2-12x}{4x^2-81}$

43. $\frac{9a^2+15a+6}{a^2+5a+4} \div \frac{36a^2-16}{3a^2+10a-8}$

44. $\frac{2x^2+5x-7}{x+4} \div \frac{x^2-2x+1}{x^2+4x}$

45. $\frac{x^2+2x}{x^2-8x} \div \frac{x+2}{3x-6}$

46. $\frac{x^3-8}{x^2-4} \div \frac{x^2+2x+4}{2x+4}$

47. $\frac{3x^2y-3xy}{3x^2+10x-8} \div \frac{4x^3y^2-4xy^2}{3x^2+x-2}$

48. $\frac{x^3-xy^2}{x^2} \div \frac{x^2+6xy-7y^2}{x^3+7x^2y}$

49. $\frac{x+4}{2x^2-14x} \cdot \frac{x^3+4x^2}{3x-24} \div \frac{x^2+8x+16}{x^2-3x-28}$

50. $\frac{4x^2-y^2}{x^2y-xy^2} \cdot \frac{x^2+xy}{8x+4y} \div \frac{2x^2-7xy+3y^2}{4x^3+2x^2y}$

51. $\frac{4x^2-20x+25}{x-3} \div \frac{2x^2-3x-5}{6x^2+15x} \cdot \frac{x^2-2x-3}{4x^2-25}$

52. $\frac{x^2-25}{x-1} \div \frac{x^2+5x}{2x^2-4x+2} \cdot \frac{x^2+2x-3}{4x^2-21x+5}$

53. $\frac{x^4-y^4}{x^2y-xy^2} \div \frac{x^2+2xy+y^2}{xy^3} \div \frac{4x^2+4y^2}{xy^2+y^3}$

54. $(25x^2-4) \div \left(\frac{x^2+2x-35}{x^2+7x} \div \frac{x-5}{5x^2+2x}\right)$