

5.1 Simplifying Rational Expressions

Now that we have mastered the process of factoring, in this chapter, we will have to use a great deal of the factoring concepts that we just learned.

We begin with the following definition

Definition: Rational Expression- an expression of the form $\frac{P}{Q}$ where P and Q are polynomials and $Q \neq 0$.

$$\frac{1}{2}, \quad \frac{x+1}{2x-1}, \quad \frac{x^2+x-1}{x+1}$$

As usual, the top is called the **numerator** and the bottom the **denominator**

So, the basic idea behind a rational expression is that a rational expression is simply a fraction which may contain variables. In this chapter, we will be learning a lot about the operations and solving of expressions and equations that contain rational expressions.

Before we get too involved in the operations of rational expressions, the first thing we need to deal with is an issue involving the denominator.

Notice that many rational expressions will be undefined at certain values, that is, where the denominator is zero. So we have to be concerned with what is called

Definition: Domain- the set of all values for which a rational expression is defined.

Dealing with the domain can be a little on the tricky side. The reason for this is that even though the domain is looking for all values for which the expression is defined, it turns out it is easier to find all the values that the rational expression is NOT defined, and simply say that you would like all "other" values.

So, that is to say, for a rational expression, the domain is all the values for which the fraction is defined. Better yet, since the only potential problem with fractions is when the denominator is zero, we can say, the domain will be all values for which the denominator is "not zero".

So, to find the domain, we find where the denominator IS zero, and then just take all "other" values.

The following examples will help with this concept.

Example 1:

Find the domain.

a. $\frac{4}{x}$

b. $\frac{x+1}{2x-1}$

c. $\frac{2x-3}{x^2+3x+2}$

Solution:

- As stated in our discussion above, to find the domain of a rational expression, it is easier to find the values for which it is not defined. This means we need to find the values for which the denominator is equal to zero.

In this case, it should be fairly easy to see that the denominator is zero only at the values when $x = 0$.

So, the domain would have to be all real numbers except $x \neq 0$.

- b. This time, we will proceed the same as we did in part a. So we need to find when the denominator of the rational expression is zero. However, this time it is not as simple as it was in part a. So, to find when the denominator is zero, we can start by setting the denominator equal to zero and then solve for x .

We get

$$\begin{aligned} 2x - 1 &= 0 \\ +1 & \quad +1 \\ \frac{2x}{2} &= \frac{1}{2} \\ x &= \frac{1}{2} \end{aligned}$$

So, this is the value that we don't want to be in the domain. Therefore, the domain is all real numbers except $x \neq 1/2$.

- c. Lastly, as we did in parts a. and b. we need to find when the denominator equals zero so that we can exclude those values from the domain. So we set the denominator to zero and solve.

We have

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ (x + 2)(x + 1) &= 0 \\ x + 2 = 0 & \quad x + 1 = 0 \\ -2 \quad -2 & \quad -1 \quad -1 \\ x = -2 & \quad x = -1 \end{aligned}$$

Since it has a squared variable, we must factor to solve

Set each factor to zero and solve

So, the domain must be all real numbers except $x \neq -2$ and $x \neq -1$.

In order to simplify matters, we tend to assume we aren't dealing with any situation where the domain in a problem. However, we will have to revisit this concept later in the chapter, and also in more depth later in this textbook.

At this point, however, we just want to be able to concentrate our efforts on just doing operations and solving equations that contain rational expressions.

The first thing we want to be able to do is reduce rational expressions. To do this we need

Fundamental Principle of Rational Expressions

$$\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q} \quad \text{and} \quad \frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}, \quad \text{where } R \neq 0$$

So, the Fundamental Principle tells us two things. First, it tells us that if we have the same thing multiplied on top and bottom of a fraction, we are allowed to cancel it out. The second thing it tells us is that we can multiply top and bottom of a fraction by any non-zero thing that we want.

We will use both parts of the Fundamental Principle, eventually, but for reducing rational expressions, we need only the first part.

Here is the process.

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| To Reduce a Rational Expression to Lowest Terms |
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|---|
| <ol style="list-style-type: none">1. Factor the numerator and denominator completely.2. Use the Fundamental Principle to cancel. |
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The basic idea is... Factor, then cancel.

Even though this seems very simple, we need to be very careful. We can **ONLY** cancel when a factor is multiplied on both the top and bottom. No other operation allows for canceling.

Example 2:

Simplify.

a. $\frac{15a^2x^3}{35ax^2}$

b. $\frac{24(x-3)}{15(x-3)}$

c. $\frac{x^2-y^2}{x-y}$

d. $\frac{3p-3q}{6q^2-6p^2}$

e. $\frac{y-4}{2y^2-7y-4}$

f. $\frac{3x^3-12x}{6x^3-24x^2+24x}$

Solution:

- a. To simplify, or in this case, reduce, a rational expression, we will follow the steps that we have been given.

In this case, we only have a monomial on top and bottom so no factoring is required. Therefore, we can just use the Fundamental Principle to cancel an a out of both top and bottom and an x^2 out of top and bottom.

Also, the 15 and the 35 reduce just like normal fractions would. It looks as follows

$$\frac{15a^2x^3}{35ax^2}$$
$$= \frac{3ax}{7}$$

- b. In this case, as in part a, we don't need to factor, simply because the rational expression is already factored completely. So we can just reduce the 24 and 15 as usual (take out a 3) and cancel the $x - 3$ from both top and bottom. We get

$$\frac{24(x-3)}{15(x-3)}$$
$$= \frac{8}{5}$$

- c. This time we will need to factor the numerator and denominator first and then cancel the common factors. Be very careful here. We must make sure we factor before we cancel. Remember, we can only cancel when we have multiplication. We might be tempted to

cancel in the beginning, but we have subtraction and not multiplication. So we proceed as follows

$$\begin{aligned} & \frac{x^2 - y^2}{x - y} \\ &= \frac{\cancel{(x - y)}(x + y)}{\cancel{x - y}} \\ &= \frac{x + y}{1} \\ &= x + y \end{aligned}$$

$x - y$ is a common factor and so it cancels

d. Again we need to start by factoring and then looking for any common factors to cancel.

$$\begin{aligned} & \frac{3p - 3q}{6q^2 - 6p^2} \\ &= \frac{3(p - q)}{6(q^2 - p^2)} \\ &= \frac{3(p - q)}{6(q - p)(q + p)} \end{aligned}$$

At this point, we should see that we have a couple of factors that are very "close" but not exactly the same. Namely, $p - q$ and $q - p$. In fact, they only differ by a negative. So if we factor a negative out of one of them, like $q - p$, then they should be the same, and then able to cancel. We get

$$\begin{aligned} & \frac{3(p - q)}{6(q - p)(q + p)} \\ &= \frac{3\cancel{(p - q)}}{-6\cancel{(p - q)}(q + p)} \end{aligned}$$

canceling common factors and reducing gives

$$= -\frac{1}{2(q + p)}$$

e. As all the other examples, lets factor and cancel. We have

$$\begin{aligned} & \frac{y - 4}{2y^2 - 7y - 4} \\ &= \frac{\cancel{y - 4}}{(2y + 1)\cancel{(y - 4)}} \\ &= \frac{1}{2y + 1} \end{aligned}$$

f. Again, we factor and cancel any common factors.

$$\frac{3x^3 - 12x}{6x^3 - 24x^2 + 24x}$$

$$\begin{aligned}
&= \frac{3x(x^2 - 4)}{6x(x^2 - 4x + 4)} \\
&= \frac{3x(x-2)(x+2)}{6x(x-2)(x+2)} \\
&= \frac{x+2}{2(x-2)}
\end{aligned}$$

As it turns out, when working with rational expressions, it is always best to simply follow the outlined procedure. Working with rational expressions by using a pre-established process is difficult enough. If we get creative with the process, not only will we get the problem wrong, but we will, in all likelihood, make a mess of the problem.

We need to keep this in mind as we work through the remainder of the chapter.

5.1 Exercises

Find the domain.

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|----------------------|--------------------------|---------------------------|-----------------------------|------------------------------|
| 1. $\frac{12}{x}$ | 2. $\frac{x+1}{x}$ | 3. $\frac{x}{x-2}$ | 4. $\frac{3x}{x-5}$ | 5. $\frac{x-4}{3x+4}$ |
| 6. $\frac{9x}{7x+1}$ | 7. $\frac{2x}{x^2-5x+6}$ | 8. $\frac{x-1}{x^2+4x+3}$ | 9. $\frac{6x+17}{2x^2+x-3}$ | 10. $\frac{3x+7}{2x^2-7x-4}$ |

Simplify.

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|-------------------------------------|-------------------------------------|---|--|
| 11. $\frac{xy^3}{x^2y}$ | 12. $\frac{x^2y}{xy}$ | 13. $\frac{5ab^3}{10ab^4}$ | 14. $\frac{12a^2b}{4a^3b}$ |
| 15. $\frac{15ay^3z}{25x^2y^2x}$ | 16. $\frac{8ab^2c^3}{36a^2bc^2}$ | 17. $\frac{18(x-4)}{24(x-4)}$ | 18. $\frac{48(x+3)}{60(x+3)}$ |
| 19. $\frac{(x-2)(x+1)}{(x-2)(x-1)}$ | 20. $\frac{(x+5)(x-3)}{(x-4)(x-3)}$ | 21. $\frac{x-2}{x^2-4}$ | 22. $\frac{a+3}{a^2-9}$ |
| 23. $\frac{x-4}{16-x^2}$ | 24. $\frac{x^3-8}{2-x}$ | 25. $\frac{20-30x}{3x^2-2x}$ | 26. $\frac{15-3x}{2x^2-10x}$ |
| 27. $\frac{x^2-1}{2x^2+x-3}$ | 28. $\frac{x^2-4x+4}{x^2-4}$ | 29. $\frac{x^2-7x+6}{x^2+3x-54}$ | 30. $\frac{2x^2+x-1}{x^2-x-2}$ |
| 31. $\frac{x^2+7x-8}{x^2+6x-7}$ | 32. $\frac{x^2+4x+4}{x^2+7x+10}$ | 33. $\frac{6-x}{x^2+3x-54}$ | 34. $\frac{5-x}{x^2-x-20}$ |
| 35. $\frac{x^3-3x^2}{2x^2-6x}$ | 36. $\frac{x^4-y^4}{x^3+xy^2}$ | 37. $\frac{2x^3-6x^2+4x}{4x^3-4x^2-8x}$ | 38. $\frac{3x^3-3x^2-18x}{6x^3-36x^2+54x}$ |
| 39. $\frac{xy-x^2}{2x^3-2xy^2}$ | 40. $\frac{ab^2-a^3}{3a^2b-3ab}$ | 41. $\frac{12a^3-12a^2b+3ab^2}{54a^4+9a^3b-18a^2b^2}$ | 42. $\frac{6x^4-96x^2}{18x^3-45x^2-108x}$ |

