4.7 Solving Equations

The question always comes up, now that we can factor… what's the point? Why do we need to do this?

The simple answer is, we use factoring to solve equations. Of course, this brings up the question of why we need to do that. However, as we have seen before, we use equations to model real world situations, i.e. we use equations to solve word problems.

We will look at the word problems in the next section. Here, we want to be able to solve equations. To do so, we will need the following property.

Zero Product Property
If $a \cdot b = 0$ then, $a = 0$ or $b = 0$.

Even though the property seems pretty obvious (if two things multiply to zero, one of them should be zero) it turns out to be a critical property in solving equations.

The reason we need the zero product property falls naturally from what we have been doing in this chapter, turning polynomials into products, also known as factoring.

Before we show an example of the zero product property in action, note that even though the property is written with only two items being multiplied, it can be extended to any number of items being multiplied together.

Let's see an example of how we use the zero product property.

Example 1:

Solve.

   a. $(x + 1)(x - 1) = 0$ b. $x(2x - 3)(4x + 1) = 0$

Solution:
   a. The first thing we should notice is that to use the zero product property to solve, we need to have multiplication involved. Here, we clearly have two binomials multiplied together and equaling zero.

   So, according to the zero product property, we can set each binomial factor equal to zero. We get
   $$(x + 1)(x - 1) = 0$$

   Now solve for x in each equation

   $$x + 1 = 0 \qquad x - 1 = 0$$
   $$ -1 \;\; -1 \qquad +1 \;\; +1$$

   $$x = -1 \qquad x = 1$$

   Since we always put solutions to equations in a solution set, our answer is $\{-1, 1\}$.

   b. Here, we again have all multiplication. So we can set each factor (GCF included) equal to zero and solve what we get.
   $$x(2x - 3)(4x + 1) = 0$$

   $$x = 0 \qquad 2x - 3 = 0 \qquad 4x + 1 = 0$$
   $$ +3 \;\; +3 \qquad -1 \;\; -1$$

$$2x = 3 \quad 4x = -1$$

$$\frac{2x}{2} = \frac{3}{2} \quad \frac{4x}{4} = \frac{-1}{4}$$

$$x = \frac{3}{2} \quad x = -\frac{1}{4}$$

So our solution set is $\left\{-\frac{1}{4}, 0, \frac{3}{2}\right\}$.

Try not to be bothered by the fact that one of the equation in the end of example 1b didn't need to have any solving done to it ($x = 0$). All that matters is that the variable is alone on one side, that makes the equation solved.

If we look closely enough at the problems in example 1, they look like what we get when we factor a polynomial completely. So, from that, we get the following process for solving.

Solving an Equation by Factoring

1. Write the equation so that everything is on one side and zero on the other.
2. Factor completely.
3. Set each factor equal to zero using the zero product property.
4. Solve each equation from step 3.
5. Check.

We should now take a look at an example of solving.

Example 2:

Solve.

    a. $x^2 - 3x - 4 = 0$ b. $2x^2 = x + 3$ c. $x^2 = 4x$ d. $y^2 = 9$

Solution:

    a. To solve, we need to follow the step outlined above. We start with noticing that equation is already written so that everything is on one side and zero on the other. So now we factor completely.

$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$

Now, as in example 1, set each factor equal to zero and solve.

$$(x - 4)(x + 1) = 0$$

$$x - 4 = 0 \quad x + 1 = 0$$
$$+4 \quad +4 \quad\quad -1 \quad -1$$

$$x = 4 \quad\quad x = -1$$

So our solution set is $\{-1, 4\}$.

    b. Here, we need to start by getting everything on one side and zero on the other.

$$2x^2 = x + 3$$
$$-x - 3 \quad -x - 3$$

$$2x^2 - x - 3 = 0$$

Now factor and set each factor equal to zero, then solve.

$$2x^2 - x - 3 = 0$$

$$(2x - 3)(x + 1) = 0$$

$$
\begin{array}{cc}
2x - 3 = 0 & x + 1 = 0 \\
+3 \quad +3 & -1 \quad -1 \\
2x = 3 & x = -1
\end{array}
$$

$$\frac{2x}{2} = \frac{3}{2}$$

$$x = \frac{3}{2}$$

So the solution is $\left\{-1, \frac{3}{2}\right\}$.

c. Again, start by moving everything to one side of the equation

$$
\begin{array}{c}
x^2 = 4x \\
-4x \quad -4x
\end{array}
$$

Factor out the GCF

$$x^2 - 4x = 0$$

Set each factor to zero

$$x(x - 4) = 0$$

Solve the equations

$$
\begin{array}{cc}
x = 0 & x - 4 = 0 \\
& +4 \quad +4 \\
& x = 4
\end{array}
$$

The solution is $\{0, 4\}$.

d. Lastly, we solve like we did on all of the problems above.

$$
\begin{array}{c}
y^2 = 9 \\
-9 \quad -9
\end{array}
$$

Move 9 to get 0 on one side

$$y^2 - 9 = 0$$

Factor the difference of squares
Set each factor to zero

$$(y - 3)(y + 3) = 0$$

Solve for x

$$
\begin{array}{cc}
y - 3 = 0 & y + 3 = 0 \\
+3 \quad +3 & -3 \quad -3 \\
y = 3 & y = -3
\end{array}
$$

So our solution is $\{-3, 3\}$.

Unfortunately, sometimes things are not as simple as those in example 2. Sometimes we need to do some additional work to get ourselves into position to actually solve. We take a look at some cases of this in the next example.

<u>Example 3</u>:

Solve.

    a. $6a^3 + 5a^2 + a = 0$ b. $y(y + 4) = 45$ c. $(z - 5)(z + 4) = 52$

Solution:

  a. Just like in example 2, we will follow the steps for solving. Notice this polynomial is a little more involved than the polynomials in example 2. However, the process should still work.

Since everything is already on one side, we just need to factor and solve.

$$6a^3 + 5a^2 + a = 0$$ Factor out the GCF

$$a(6a^2 + 5a + 1) = 0$$ Factor the trinomial

$$a(3a + 1)(2a + 1) = 0$$ Set each factor to zero

$$a = 0 \quad 3a + 1 = 0 \quad 2a + 1 = 0$$
$$ -1 \quad -1 \quad\quad -1 \quad -1$$ Solve for a

$$3a = -1 \quad\quad 2a = -1$$

$$\frac{3a}{3} = \frac{-1}{3} \quad\quad \frac{2a}{2} = \frac{-1}{2}$$

$$a = -\frac{1}{3} \quad\quad a = -\frac{1}{2}$$

So, this time we have three solutions. It doesn't matter how many solutions we have, we simply put them into a solution set. This means the solution is $\left\{-\frac{1}{2}, -\frac{1}{3}, 0\right\}$.

  b. This time, the equation looks very different then all of the others. It almost looks like the equation has already been factored. However, in order to use the <u>zero</u> product property, we need to have just that…. A zero on one side.

So, the best course of action is to multiply out the left side, then get the zero on one side and take it from there.

$$y(y + 4) = 45$$ Multiply out the left side

$$y^2 + 4y = 45$$
$$-45 \quad -45$$ Get 0 on one side

$$y^2 + 4y - 45 = 0$$ Factor

$$(y + 9)(y - 5) = 0$$ Set each factor to zero

$$y + 9 = 0 \quad\quad y - 5 = 0$$
$$-9 \quad -9 \quad\quad +5 \quad +5$$ Solve

$$y = -9 \quad\quad y = 5$$

So the solution is $\{-9, 5\}$.

c. Again, this is similar to part b above. We need to multiply out the two binomials as we learned in chapter 3, then proceed as usual.

$$(z - 5)(z + 4) = 52$$

Multiply out the binomials

$$z^2 + 4z - 5z - 20 = 52$$

Combine like terms

$$z^2 - z - 20 = 52$$
$$ -52 -52$$

Get 0 on one side

$$z^2 - z - 72 = 0$$

Factor

$$(z - 9)(z + 8) = 0$$

Set each factor to zero

$$z - 9 = 0 \quad z + 8 = 0$$
$$+9 +9 -8 -8$$

Solve

$$z = 9 \quad z = -8$$

So the solution set is $\{-8, 9\}$.

4.7 Exercises

Solve.

1. $(x - 3)(x + 2) = 0$

2. $(y - 4)(y + 7) = 0$

3. $(2x - 3)(5x - 1) = 0$

4. $(7x + 3)(x - 4) = 0$

5. $a(a - 9)(a + 3) = 0$

6. $b(b - 2)(b + 2) = 0$

7. $2x(x - 2)(2x + 1) = 0$

8. $5a(2a + 1)(6a - 5) = 0$

9. $(x - 2)(5x + 2)(x - 3) = 0$

10. $y^2 + 7y + 12 = 0$

11. $x^2 - 12x + 27 = 0$

12. $12x^2 + 29x - 8 = 0$

13. $4y^2 + 31y - 45 = 0$

14. $a^2 + 4a = 0$

15. $4x^2 - 1 = 0$

16. $u^2 = 5u + 14$

17. $2t^2 - 4t = t + 3$

18. $x^2 - 2x - 19 = 2x + 13$

19. $2x^2 + 2x = 7x + 12$

20. $p^2 + 6p = 6p + 1$

21. $6z^2 + z = -10z - 3$

22. $a^3 = a$

23. $b^2 = 3b$

24. $x^2 = 4x - 4$

25. $z^3 = 16z$

26. $x^2 - 3x - 9 = 3x + 7$

27. $a^2 = 100$

28. $b^2 - 3b = 2b + 14$

29. $3x^2 = 6 - 3x$

30. $4a^2 + 33a = 36 + a$

31. $3x^3 + 16x^2 = 35x$

32. $2x^3 = 4x^2 - x$

33. $x^3 = 4x$

34. $a(a - 6) = 7$

35. $x(x - 14) = 15$

36. $t(t - 8) = 9$

37. $b(b - 5) = 36$

38. $x(x + 8) = 9$

39. $(x + 1)(x - 2) = 4$

40. $(k + 1)(k - 2) = 28$

41. $(a + 4)(a - 6) = -21$

42. $x(x + 5) = 5(x + 5)$

43. $x(x + 6) = 6(x + 6)$

44. $x(x + 4) = 4(x + 9)$

45. $t + 18 = t(t + 8)$

46. $3z(z + 5) = 10(z + 5)$ 47. $x + 3 = x(x + 3)$ 48. $(x - 3)^2 = 4$

49. $x(x + 2) = 2(x + 8)$ 50. $(x + 6)(x + 5) = 2$ 51. $(x - 1)(x + 5) = -8$

52. $(x - 1)(5x + 4) = 2$ 53. $(x + 1)^2 = x + 7$ 54. $(x + 3)(3x + 5) = 7$

55. $(x - 2)(6x - 1) = -4$ 56. $(x - 7)(x + 1) + 16 = 0$ 57. $(t - 2)(t + 5) = 18$

58. $(x + 2)(x - 7) = -18$ 59. $(r + 1)^2 - r = 7$ 60. $(x + 3)(2x - 1) = 9$

61. $x^2(2x - 1) = 3x$ 62. $(x - 2)^2 = -4(x + 2)$ 63. $x^2 = x(10 - 3x^2)$