4.6 General Factoring Strategy

Now that we have mastered all of the different factoring techniques, how do we know which one to use to factor a given polynomial?

It turns out, the answer to this questions is contained entirely within the way in which we learned the techniques. We learned each technique (except the GCF) by the number of terms the polynomial contained.

So, we will choose our strategy for factoring based on the number of terms. Here is the idea

General Strategy for Factoring

1. Factor out the GCF.
2. Count the number of terms in the remaining polynomial. If it has
    a. Four terms- Factor by grouping
    b. Three terms- Factor by trial factors
    c. Two terms- Factor by factoring formulas
3. Check each factor to see if you can factor it further. If so, then we factor again.

The concept behind step 3 above, is that of factoring completely. This means, we cannot leave any factor that could be factored again. We saw some of this in the last section, but we will see much more of it in this section.

One way that we can tell if we have factored completely is by simply looking at what we got and deciding if it can be factored further. We do, however, have a few helpful tips.

1. A binomial is factored completely if it contains any variables to the first power.
2. Any trinomial that comes from the sum or difference of squares formula cannot be factored.
3. The sum of squares does not factor.

Beyond this, you would have to check the factors individually to see if they can be factored further.

Before beginning with the general factoring we also want to mention that factoring completely is not a "new" topic. It is simply the necessary consequence of learning all the techniques we have in this chapter. So as long as we have truly mastered each technique individually, we should be able to factor any polynomial, with four terms or less, completely.

Now let's apply the technique to some examples.

Example 1:

Factor completely.

a. $5x^2 - 32x - 21$ b. $a^3 + a^2b + ab^2 + b^3$ c. $8b^3 - c^3$

Solution:

a. To factor completely, we begin by looking for a GCF. Since this does not have a GCF, we next count the terms. Here, we have 3 terms which means we need to factor by trial factors as we learned in section 4.4. We get

$$5x^2 - 32x - 21$$
$$= (5x + 3)(x - 7)$$

Since each of the binomials contains a first power variable, they cannot be factored further. Therefore, we have factored completely.

b. Again, we notice that we do not have a GCF. Since we have 4 terms here, we will have to factor by grouping, as we did in section 4.2.

$$\begin{aligned} &a^3 + a^2b + ab^2 + b^3 \\ &= a^2(a + b) + b^2(a + b) \\ &= (a + b)(a^2 + b^2) \end{aligned}$$

Since the first binomial has first power variables, and the second binomial is the sum of squares, neither of them factor more. Thus, we have factored completely.

c. Again, we do not have a GCF. So, here we have 2 terms. This means we need to factor it by using the formulas, and technique we learned in 4.5. We clearly have the difference of cubes, with our $a = 2b$ and $b = c$. So we get

$$\begin{aligned} &8b^3 - c^3 \\ &= (2b)^3 - (c)^3 \\ &= (2b - c)((2b)^2 + 2b \cdot c + c^2) \\ &= (2b - c)(4b^2 + 2bc + c^2) \end{aligned}$$

Since the binomial has a first power variable, and the trinomial came from the difference of cubes formula, they do not factor. So, we have factored completely.

Notice all of the problems in example 1 simply factored once and then were finished. In example 2, we will see polynomials that require more than just one step of factoring. That is, they will need to be factored completely.

<u>Example 2</u>:

Factor Completely.

a. $12x^3y^2 - 38x^2y^3 + 16xy^4$ b. $x^4 - 81$

c. $x^5 - 4x^3 - 8x^2 + 32$ d. $128 - 2y^6$

Solution:

a. The first step in factoring is always, factor out the GCF. In this case, the GCF is $2xy^2$.

$$\begin{aligned} &12x^3y^2 - 38x^2y^3 + 16xy^4 \\ &= 2xy^2(6x^2 - 19xy + 8y^2) \end{aligned}$$

Now we factor the remaining trinomial by trial factors.

$$\begin{aligned} &2xy^2(6x^2 - 19xy + 8y^2) \\ &= 2xy^2(3x - 8y)(2x - y) \end{aligned}$$

Since each binomial contains at least one first power variable, the polynomial is completely factored.

b. First, this binomial does not have a GCF. So, we need to factor by formula. Here, we have a difference of squares. We factor as follows

$$\begin{aligned} &x^4 - 81 \\ &= (x^2)^2 - (9)^2 \\ &= (x^2 - 9)(x^2 + 9) \end{aligned}$$

Now, we see if each factor can be factored further. It should be fairly clear that the first binomial is, again, a difference of squares. Therefore, it can be factored again. The back binomial is a sum of squares and therefore cannot be factored. So, factoring the front we get

$$(x^2 - 9)(x^2 + 9)$$
$$= (x - 3)(x + 3)(x^2 + 9)$$

Now the polynomial is completely factored.

c. Again, we do not have a GCF to factor out. So, this time, we have a four term polynomial. This means, we must use the grouping method to factor.

$$x^5 - 4x^3 - 8x^2 + 32$$
$$= x^3(x^2 - 4) - 8(x^2 - 4)$$
$$= (x^2 - 4)(x^3 - 8)$$

Each of these resulting binomials can be factored further. The front binomial is a difference of squares, and the back binomial is a difference of cubes. So we will factor by formulas as we usually do.

$$(x^2 - 4)(x^3 - 8)$$
$$= (x - 2)(x + 2)(x - 2)(x^2 + 2x + 4)$$
$$= (x - 2)^2(x + 2)(x^2 + 2x + 4)$$

Since the binomials have first power variables and the trinomial came from the difference of cubes, we have factored completely.

Notice we still condense the repeated binomials into a square. We always want to make sure and do this step to finish up.

d. Lastly, we need to start by factoring out the GCF. Since the leading term is negative, we will also factor out a negative and rearrange the order of the terms.

$$128 - 2y^6$$
$$= -2(y^6 - 64)$$

In this case, the factoring is quite difficult. The reason is, this binomial is both a difference of squares (64 is 8^2) and a difference of cubes (64 is 4^3). So, as we mentioned in the end of the last section, this means we need to do the difference of squares first. This gives us

$$-2(y^6 - 64)$$
$$= -2(y^3 - 8)(y^3 + 8)$$

Now checking for more factoring shows that we now have a difference of cubes and a sum of cubes. So we use our formulas to factor.

$$-2(y^3 - 8)(y^3 + 8)$$
$$= -2(y - 2)(y^2 + 2y + 4)(y + 2)(y^2 - 2y + 4)$$

Since each binomial has first power variables, and since each trinomial came from the sum or difference of cubes, none of them factor. This means, we have completely factored.

Keep in mind, factoring completely is nothing new. It is simply an extension of using the factoring techniques over and over again, until no more factoring can be done.

4.6 Exercises

Factor Completely.

1. $5y^2 - 32y - 21$	2. $72x^2 + 4x - 8$	3. $x^3 - 8$
4. $a^2 - 10a - 144$	5. $12x^2 - 16xy - 3y^2$	6. $y^2 - 100$
7. $8x^3 - 4x^2 + 6x - 3$	8. $x^3 - 64$	9. $4x^2 - 81$
10. $x^3 - 8x^2 - 3x + 24$	11. $14x^2 - 41x + 15$	12. $p^3 + 1$
13. $4x^3 + 8x^2 - 3x - 6$	14. $6x^2 + 29xy - 42y^2$	15. $a^3 + 27$
16. $x^8 - 81y^6$	17. $27x^3 + 8y^3$	18. $15y^2 - 55y + 40$
19. $20 + 5a^2 - 4a - a^3$	20. $x^3 - x^2 - 3x + 3$	21. $5x^4 - 30x^3 + 40x^2$
22. $15x^2y - 27xy - 6y$	23. $n^4 - 1$	24. $x^3 + 6x^2 + xy + 6y$
25. $20x^5 + 28x^4 - 24x^3$	26. $a^2b^3 - 2ab^2 - 24b$	27. $b^4 - 16$
28. $8x^3 - 27y^3$	29. $a^4 - 12a^2 + 27$	30. $25x^3 - 30x^2 + 9x$
31. $5a^3b + 30a^2b - 80ab$	32. $5x^2 + 25x - 120$	33. $3x^3 + 21x^2 + 30x$
34. $2ab^2 + 28ab + 80a$	35. $u^4 + u^3 - 56u^2$	36. $x^2a^2 - x^2b - a^2y + yb$
37. $x^2 + 5x + xy + 5y$	38. $2ab^2 - 30ab + 108a$	39. $36x^3 - 64x$
40. $b^4 + 16b^3 + 64b^2$	41. $x^2 - xd + 7x - 7d$	42. $14ab^2 - 11ab + 2b$
43. $35x^2 - 100x - 15$	44. $64x^4y^4 - 8xy$	45. $xy + 8x - y^2 - 8y$
46. $4x^8 - 16x^7 + 16x^6$	47. $x^3 + 8y^3$	48. $20x^4y^4 - 73x^3y^3 + 63x^2y^2$
49. $2ax^2 - 22ax + 60a$	50. $2x^4 - 16x$	51. $x^4 - y^4$
52. $4x^2y + 56xy + 192y$	53. $x^4 - 29x^2 + 100$	54. $-3n^2 - 12nm + 15m^2$
55. $8x^4 + 56x^3 + 98x^2$	56. $27a^3x^2 - 64x^2b^3$	57. $3n^2m - 17nm + 24m$
58. $2x^2y^4 - 17xy^3 - 460y^2$	59. $4x^3y - 49xy^3$	60. $81abc^2 - 100ab^3$
61. $5x^2 + 20xy - 60y^2$	62. $w^4 - 625$	63. $6x^3 - 2x^2y + 24x - 8y$
64. $n^4 - 13n^2 + 36$	65. $2x^4y - 3x^3y - 20x^2y$	66. $7uv^3 - 252uv$
67. $9x^3y + 33x^2y^2 + 30xy^3$	68. $2rs^4t - 8r^3s^8t$	69. $3x^{19}y^9 - 81xz^3$
70. $50x^2y - 162y^9$	71. $6a^3b^4 + 40a^2b^5 + 8ab^3$	72. $a^6b^9 - c^{30}$

73. $t^4 - 37t^2 + 36$

74. $60a^2 + 38ab - 126b^2$

75. $2a^7b^3 - 288ab$

76. $4x^3 - 8x^2 - 25x + 50$

77. $x^5 - 9x^3 - x^2 + 9$

78. $p^4 - 10p^2 + 9$

79. $x^9y^{12} + z^{15}$

80. $343x^3y^9 + 64z^{12}$

81. $6a^4b^2 - 11a^3b^3 + 4a^2b^4$

82. $z^8 - 1$

83. $x^6y - y$

84. $a^7 - a$

85. $x^4 + 2x^3 - 4x^2 - 8x$

86. $x^4y + 2x^3y + 27xy + 54y$