4.5 Factoring Binomials

The last type of factoring that we need to look at is factoring binomials. Once we are able to factor those, we will have to discuss how to determine which technique to use on a given polynomial. We will discuss this in the next section.

In the meantime, the only thing you need to know in order to factor binomials is that it's all about using formulas. There are only 3 type of binomials that we will look at here.

The first one of them comes from one of the special products in section 3.4. The other formulas are new. We will give them without proof of their validity, however, it is relatively easy to multiply them out to see that they are, in fact, true.

Here are the binomial formulas:

Binomial Factoring Formulas
$a^2 - b^2 = (a - b)(a + b)$ Called the <u>Difference of Squares</u> $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Called the <u>Difference of Cubes</u> $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Called the <u>Sum of Cubes</u>

Before we proceed to some examples, notice that there is no "Sum of Squares" formula. That is because the sum of squares never EVER factors. If you happen to run across a sum of squares, simply leave it alone and move on with the problem.

Also, always keep in mind that factoring binomials is all about the formulas, period. So, since this is the case, you must memorize these three formulas if you want to be successful in factoring binomials.

Lastly, its important to note that the trinomial part of the sum and difference of cubes does not factor any further. We will discuss this idea of factoring completely later in this sections and in the next section, but for now it is enough to say that whenever we use one of those two formulas, the trinomial part is finished with factoring.

All this being said, let's move to some examples.

<u>Example1</u>:

Factor.

    a. $x^2 - 25$ b. $t^3 + 8$ c. $p^3 - q^3$ d. $4y^2 - 9$

Solution:
    a.   The first thing we need to do to factor a binomial is determine which of the three types it is. Usually it is fairly easy since the name corresponds exactly with what they contain.

           In this case, we clearly have a difference of squares, that is, since the x is squared and we know that 25 is 5^2. So, we might want to "look" at our binomial as
$$x^2 - 25 = x^2 - 5^2$$

With this in mind, the most important thing we need to know when using the formulas to factor, is that in the formula, we actually use whatever it is that is being squared (or cubed). This means, we actually use the x and the 5 as our a and b. So, here, $a = x$ and $b = 5$. Plugging them into our formula gives us

$$x^2 - 5^2$$
$$= (x - 5)(x + 5)$$

$$a^2 - b^2$$
$$= (a - b)(a + b)$$

b. This time, we are working with a sum of cubes. This is really easy to see since our t is cubed and we also need to know that 8 is 2^3. So, in this case, since the values we work with in the formulas are the values that are being cubed (namely the t and the 2) our $a = t$ and $b = 2$ in the formula. So, plugging in these values gives us

$$t^3 + 8$$
$$= t^3 + 2^3 \qquad \text{Using}$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$= (t + 2)(t^2 - 2t + 2^2)$$
$$= (t + 2)(t^2 - 2t + 4)$$

c. This time we are obviously working with a difference of cubes. Here, our $a = p$ and our $b = q$. So we substitute those values into the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ as follows

$$p^3 - q^3$$
$$= (p - q)(p^2 + pq + q^2)$$

d. Finally, this example is a little more challenging. We should be able to tell that we are working with a difference of squares. But, the first thing we need to do, is to be able to tell what is, in fact, being squared in each term.

We know that 4 = 2^2 and 9 = 3^2. So we can rewrite our binomial as
$$4y^2 - 9$$
$$= (2y)^2 - (3)^2$$

So our $a = 2y$ and $b = 3$. Putting them into the formula gives us
$$(2y)^2 - (3)^2$$
$$= (2y - 3)(2y + 3)$$

Now that we have the basic idea, let's take a look at some others.

Example 2:

Factor.

a. $64a^3 - 8$ b. $2x^2y^3 - 8y$ c. $a^{15}b^{12} - c^6$ d. $x^4 - 1$

Solution:
a. As is all factoring, the first thing we need to do is factor out the GCF. Here the GCF is 8. We get
$$64a^3 - 8$$
$$= 8(8a^3 - 1)$$

Now, we have the difference of cubes. Keep in mind that 1 raised to any power still gives us 1. Therefore, 1 can be whatever power we need it to be. In this case, we need it to be a third power. So, we can rewrite our binomial (so that we can identify our "a" and "b" for the formula) as
$$8((2a)^3 - (1)^3)$$

So, our $a = 2a$ and our $b = 1$. Plugging them into $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ gives

$$8((2a)^3 - (1)^3)$$
$$= 8(2a - 1)((2a)^2 + 2a \cdot 1 + 1^2)$$

Now we simply clean it up a bit by multiplying and working out the powers.

$$8(2a - 1)((2a)^2 + 2a \cdot 1 + 1^2)$$
$$= 8(2a - 1)(4a^2 + 2a + 1)$$

Lastly, we should always make sure none of the polynomials that remain are able to be factored. This is the idea of <u>factoring completely</u>. If something can be factored, it must be factored.

In this case, the only thing that might be able to be factored is the trinomial. However, as we said before we started example 1, the trinomial part of a sum or difference of cubes formula does not factor. Therefore, the problem is completely factored.

b. Again, we need to start with factoring out the GCF.

$$2x^2y^3 - 8y$$
$$= 2y(x^2y^2 - 4)$$

We are left with a difference of squares. So we factor accordingly. We have

$$2y(x^2y^2 - 4)$$
$$= 2y((xy)^2 - 2^2) \qquad \text{a = xy, b = 2}$$
$$= 2y(xy - 2)(xy + 2) \qquad \text{Into the formula } a^2 - b^2 = (a - b)(a + b)$$

c. This time, we do not have a GCF. Also, this time it is particularly difficult to determine what type of binomial we have. The easy way to decide is by looking at the powers.

Here, each of the powers is divisible by 3. So, it must be a difference of cubes problem. We will verify by making each term look like a cube. This will also tell us what our "a" and "b" are for the formula.

$$a^{15}b^{12} - c^6$$
$$= (a^5b^4)^3 - (c^2)^3$$

So our $a = a^5b^4$ and $b = c^2$.
Plugging them into the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ gives

$$(a^5b^4 - c^2)((a^5b^4)^2 + a^5b^4 \cdot c^2 + (c^2)^2)$$

Simplifying gives

$$(a^5b^4 - c^2)(a^{10}b^8 + a^5b^4c^2 + c^4)$$

Now, we check for more factoring. The binomial in front is not a difference of squares or cubes, and the trinomial came from a cubes formula. Therefore, the binomial has been completely factored.

d. Lastly, since we have no GCF, we need to determine what type of binomial we have. Like in part c above, we have to look at the powers. Clearly, the power of x is only divisible by 2, and since the other term is a 1, we must have the difference of squares.

So we can rewrite our binomial as

$$x^4 - 1$$
$$= (x^2)^2 - 1^2$$

So, our $a = x^2$ and our $b = 1$. Using the formula $a^2 - b^2 = (a - b)(a + b)$ gives

$$(x^2)^2 - 1^2$$
$$= (x^2 - 1)(x^2 + 1)$$

Now, we should notice that we again have a difference of squares (x^2 - 1^2) and a sum of squares, but recall that a sum of squares will never factor. But we do need to factor the difference of squares, i.e. the front binomial. We get

$$(x^2 - 1)(x^2 + 1)$$
$$= (x - 1)(x + 1)(x^2 + 1)$$

Since each binomial is factored completely, this is our answer.

Note: If you are trying to factor a binomial and you have the option of both the difference of squares and the difference of cubes, **always factor the difference of squares first**. This will lead you to a difference and sum of cubes that you will need to factor at that point.

4.5 Exercises

Factor.

1. $x^2 - 4$

2. $a^2 - 9$

3. $x^3 - y^3$

4. $x^3 - 8$

5. $a^3 + 27$

6. $b^3 - 1$

7. $t^2 + 9$

8. $n^3 + m^3$

9. $y^3 - 1$

10. $u^2 - v^2$

11. $b^2 - 16$

12. $x^2 + 1$

13. $x^3 - 8y^3$

14. $4x^2 - 9y^2$

15. $64c^3 + 27$

16. $a^3 + 27b^3$

17. $x^2 - 81y^2$

18. $8x^3 - 1$

19. $4x^2 - 1$

20. $x^2 - 64y^2$

21. $x^3y^3 + z^3$

22. $a^3b^3 - c^3$

23. $4a^2 - 121b^2$

24. $x^3 - 8y^3z^3$

25. $8x^3 - 125y^3z^3$

26. $x^6 + 64y^6$

27. $x^6y^3 + z^9$

28. $p^6 - 27q^9n^3$

29. $a^2b^2 - c^4d^4$

30. $a^2b^8 - 4c^4$

31. $8n^9 - 27m^{12}$

32. $x^9y^{12} + n^{12}m^{15}$

33. $x^3y^{12} + a^9b^{18}$

34. $4x^3 + 32y^6$

35. $2x^4 - 8$

36. $ab^4 + ab$

37. $8p^6 - 8q^9$

38. $n^2m - 9m^5$

39. $x^4y^7 + xy^4$

40. $c - b^3c^{10}$

41. $8 - 2a^2b^8$

42. $a^4 - b^4$

43. $xy^5z^6 - x^4y^2$

44. $4xy^3 + 32xy^6$

45. $x^4 - 81$

46. $a^5b^7 - a^2bc^{12}$

47. $a^6b^3c^9 - 27$

48. $4a^4b^4 - 64$

49. $16xy^6 + 54x^4$

50. $8ab^4 - a^4b$

51. $27x^3y^3 - 3x^3y$

52. $48x^4y^7 + 6xy$

53. $81x^4 - 24xy^6$

54. $a^6 - b^6$

55. $x^8 - y^8$

56. $x^{16} - y^{16}$

57. $x^9 - y^9$

58. $x^9 + y^9$

59. $2x^6 - 2$

60. $a^6 - 64$