4.4 Factoring $ax^2 + bx + c$

From the last section, we now know a trinomial should factor as two binomials.

With this in mind, we need to look at how to factor a trinomial when the leading coefficient is something other than 1.

As we mentioned in the beginning of the last section, there are other techniques that we could use to factor trinomials, such as using x's, boxes, grouping, etc, but in order to simplify matters, we have chosen to only use the trial factors method. It is generally not a very good idea to provide numerous methods for solving the same problems. It tends to cause confusion of techniques.

With this being said, the basic idea will remain the same as it was in the last section. We will use trial factors.

The only difference here is that instead of having to choose values for just the back terms of the binomials, we will also have to choose terms for the front binomials.

The idea is actually quite simple. Knowing that the binomials would need to multiply back out to the original trinomial, clearly the front terms must always multiply to become the front term and the back terms must always multiply to become the back term.

Therefore, we will need to only check the middle term if we set the binomials up so that the "fronts make the front and the backs make the back".

We state it like this

Factoring by Trial Factors

multiply to c

$$ax^2 + bx + c = (\_\_\_\_x + \_\_\_\_)(\_\_\_\_x + \_\_\_\_)$$

multiply to a

We make a trial "guess" at values for the front and back and then check to see if the middle term works by multiplying the binomials out.

Also, we will need to keep in mind all of the rules that we had for the signs from the last section.

Rules for Signs in Trial Factors

1. If the trinomial has all positive signs then the binomials it factors into will have all positive signs, that is, **all positive means all positive**.
2. If the trinomial has a last term that is positive and the middle term negative, then both binomials it factors into will both have negatives, that is, **if the back is positive and the middle is negative, both binomials are negative**.
3. If the trinomial has a last term that is negative, regardless of the middle term, then one binomial is positive and one is negative, that is, **if the back is negative, then one is positive and one is negative**.

Using these rules and keeping in mind that we simply need to make a "guess" and check to see if our guess was correct by multiplying, we can take a look at some examples.

Example 1:

Factor.

a. $4x^2 + 15x + 9$ b. $2x^2 - 11x + 12$ c. $3a^2 + 8a - 4$

Solution:

a. To factor this trinomial, we use the trial factors method. As stated above, we will need to choose values for the front of each binomial so that they multiply to the front of our original trinomial, so in this case our options are 1 and 4, or 2 and 2. And our back terms of our binomials must multiply to the back term of our original trinomial, meaning our choices are 1 and 9 or 3 and 3. Lets start by trying the 2 and 2 and the 3 and 3. Remember, since everything is positive, the binomials are all positive.

$$4x^2 + 15x + 9$$
$$= (2x + 3)(2x + 3)$$

Now, we need to multiply it out to see if we are correct. Before we do so, remember that the front term is set up by design so that it will become the front of the trinomial, and the back term is set up by design to produce the back term of the trinomial. Therefore, we need to only check the middle term.

The middle term of a factored trinomial always comes from multiplying the front of the first binomial times the back of the second added to the back of the first times the front of the second. So we check accordingly.

$$(2x + 3)(2x + 3)$$
$$6x + 6x = 12x$$

Since we didn't get the correct middle term (15x), this is not factored correctly. This means we need to try a different cominition of our trial values. We suggest that you don't try to change everything all at once. Just change only the front's or the back's and try again. Let's try

$$4x^2 + 15x + 9$$
$$= (4x + 3)(x + 3)$$

This time checking the middle term like we did above gives 12x + 3x = 15x. Since this is the correct middle term, we have factored correctly.

b. Again, we use trial factors. Our only choice for the front term are 1 and 2, which is nice because it means we will have less options for factoring. Our choices for the back terms are 1 and 12, 2 and 6 or 3 and 4.

By our rule for signs, both binomials will have a negative. Now, in deciding which options to try first, keep in mind that our origonal trinomial does not have a GCF. Therefore, none of the binomials can have a GCF because if they did, the origional would have had to have that same GCF. So, in choosing where to put things, you should not have a 2 in the same binomial as another 2, or 6, or 12 etc.
This will help to limit the number of trials we will need to make on our factoring. So, lets try 1 and 2 for the fronts, and 3 and 4 for the backs (making sure the 4 is not with the 2).

$$2x^2 - 11x + 12$$
$$= (2x - 3)(x - 4)$$

Now checking the middle term (front of first time back of second plus back of first times front of second) we get -8x + -3x = -11x. Which is correct. Thus, we have factored correctly.

c. Again, we will use trial factors. Since our front term is 3, our only choice is 3 and 1. For the back, we can try 1 and 4 or 2 and 2. Also, since the back term is negaitive, we (as we did in the last section) can put a positive and negative wherever we want and if they are wrong, we will know when we check. Let's try the following

$$3a^2 + 8a - 4$$
$$= (3a - 2)(a + 2)$$

Checking the middle term we get 6a + -2a = 4a which is not correct. So lets try

$$3a^2 + 8a - 4$$
$$= (3a - 4)(a + 1)$$

Checking gives 3a + -4a = -a. also not correct. Just because these didn't work, we are not yet out of options. We can still try changing where we put the 1 and 4. When factoring trinomials with the leading coefficient not 1, the location of our trial numbers can, and will, change everything. So we need to make sure we try switching spots as well.

$$3a^2 + 8a - 4$$
$$= (3a - 1)(a + 4)$$

Checking this gives 12a + - a = 11a. This doesn't work either. Since we are completely out of options, The trinomial must be prime.

Now that we have the basic idea, let's start working on some trinomials that are tougher and trickier.

Example 2:

Factor.

a. $2x^3 - x^2 - x$ b. $8y^2 - 8y - 6$ c. $18 - 17x - x^2$

d. $6x^2y - 19xy + 10y$ e. $4x^3y + 18x^2y - 36xy$ f. $36x^2y - 48xy^2 + 16y^3$

Solution:

a. First, we need to notice that we have a GCF of x. As always, we need to factor that out first. This gives,

$$2x^3 - x^2 - x$$
$$= x(2x^2 - x - 1)$$

Now factor with trial factors. We have very few choices because of the numbers we have involved. Really we must use 1 and 2 in front, and 1 and 1 in the back.

$$x(2x^2 - x - 1)$$
$$= x(2x - 1)(x + 1)$$

Checking gives 2x + -x = x. Since this is only off by a sign, we only need to swap our positive and negative and it will be correct. We get

$$2x^3 - x^2 - x$$
$$= x(2x + 1)(x - 1)$$

b. Again, we start with the GCF.

$$8y^2 - 8y - 6$$
$$= 2(4y^2 - 4y - 3)$$

Now use trial factors

$$2(4y^2 - 4y - 3)$$
$$= 2(2y - 3)(2y + 1)$$

Checking the middle term gives 2y + -6y = -4y. So our trinomial is factored correctly.

c. This time, we should see that the trinomial is set up a little strangely. The first thing we should probably do is rewrite it in descending order.

$$18 - 17x - x^2$$
$$= -x^2 - 17x + 18$$

As a piece of advice, it is always easier to factor a trinomial when the leading coefficient is positive. So, in this case, it would likely be a good idea of factor the negative out before attempting to continue. Remember, factoring out a negative will switch all of the signs. We get

$$-x^2 - 17x + 18$$
$$= -(x^2 + 17x - 18)$$

Now we can factor with trial factors.

$$-(x^2 + 17x - 18)$$
$$= -(x + 18)(x - 1)$$

Checking the middle term shows that this is correctly factored.

d. Again, we need to start by factoring out the GCF.

$$6x^2y - 19xy + 10y$$
$$= y(6x^2 - 19x + 10)$$

Now, we need to factor by trial factors.

In this case, we have numerous options for coefficients for the front and back. The options for the front are 1 and 6, or 2 and 3, the options for the back are 1 and 10, or 2 and 5. This means we have 8 different potential choices for our binomials. However, keep in mind that we will not have a GCF within any binomial, which makes our choices less than it seems like.

Nevertheless, even though this seems like a lot of options, we simply need to keep "tinkering" with all of them until we find the choice that produces the correct middle term. As it turns out the correct factoring is

$$y(6x^2 - 19x + 10)$$
$$= y(2x - 5)(3x - 2)$$

e. We, again, start with factoring out a GCF.

$$4x^3y + 18x^2y - 36xy$$
$$= 2xy(2x^2y^2 + 9xy - 18)$$

Now that the GCF is out, we should notice that the front term has x^2y^2. As we saw in the last section, this simply means that we will need an xy in each of the front terms.

Fortunately, we don't have many choices for the front numbers (only 1 and 2 is an option), but we do have several choices for the back. Also, remember the rules for signs, one must be positive and one must be negative.

Factoring by trials gives us

$$2xy(2x^2y^2 + 9xy - 18)$$
$$= 2xy(2xy - 3)(xy + 6)$$

f. Lastly, start with the GCF, then factor by trial factors, being careful with variables and signs. We get

$$36x^2y - 48xy^2 + 16y^3$$
$$= 4y(9x^2 - 12xy + 4y^2)$$
$$= 4y(3x - 2y)(3x - 2y)$$
$$= 4y(3x - 2y)^2$$

Be aware that the trial factors method can require a great number of trials to be successful. Try not to get too frustrated. Trial factors is the kind of process that we will get better at the more we practice it.

We simply need to commit to keep plugging away at all of the options until we find the one that work, unless, of course, the trinomial happens to be prime.

4.4 Exercises

Factor.

1. $3x^2 + 8x + 5$
2. $6a^2 - 11a - 10$
3. $4x^2 + 21x + 5$
4. $10x^2 - 27x + 5$
5. $5y^2 + 17y - 40$
6. $2x^2 - 5x + 6$
7. $9x^2 + 11x + 4$
8. $3u^2 + 2u - 21$
9. $12x^2 + 13xy - 35y^2$
10. $15x^2 - 14x + 3$
11. $10x^2y^2 + xy - 3$
12. $20b^2 + 13b - 15$
13. $6a^2 - 7a + 2$
14. $4x^2 - 40xy + 25y^2$
15. $35x^2 - 54x + 7$
16. $12x^2 - 7xy - 12y^2$
17. $9x^2 - 30x + 25$
18. $14y^2 - 51y + 7$
19. $9n^2 + 11n + 4$
20. $20m^2 + 11m - 3$
21. $4a^2 + 9ab - 28b^2$
22. $7c^2 + 12c - 15$
23. $12n^2m^2 - 32nm - 35$
24. $6p^2 - 11pq + 3p^2$
25. $16x^2 - 24x + 9$
26. $9a^2b^2 - 24ab + 16$
27. $56x^2 + 121x + 63$
28. $35x^2 - 32x + 5$
29. $15p^2 + 2pq - 8q^2$
30. $56a^2 + 17ab - 3b^2$

31. $15x^2 - 10x - 40$

32. $15x^2 - 35x - 30$

33. $8x^2 - 80x + 200$

34. $420 + 115x - 25x^2$

35. $-18x^2 + 30x + 672$

36. $15x^2y^2 - 25xy - 60$

37. $20ab^2 + 37a^2b + 15a^3$

38. $3x^3 + 2x^2 - 21x$

39. $9x^4y^2 + 24x^3y + 15x^2$

40. $16x^2 - 96x + 144$

41. $60y + 25xy - 15x^2y$

42. $125x^2 - 100x + 20$

43. $8x^2y^2 - 40xy^2 + 50y^2$

44. $12xy^2 + 13xy - 35x$

45. $12L^3W^3 - 41L^2W^2 + 24LW$

46. $20x^2z - 6xz - 36z$

47. $32bc^2 + 28bc - 30b$

48. $45ax^2 - 42axy + 9ay^2$

49. $9x^3y - 15x^2y^2 - 36xy^3$

50. $12a^2b^3 + 8ab^2 - 84b$

51. $70u^2v - 108uv + 14v$

52. $12x^2y - 76xy - 616y$

53. $13ab^2 - 78ab + 117a$

54. $63a^2 - 23a^2b - 56a^2b^2$

55. $-12x^3y^2 + 26x^2y - 12x$

56. $9x^2y - 54xy + 81y$

57. $16x^2y + 78xy + 27y$

58. $40uv^2 + 38uv + 6u$

59. $80x^3z^2 - 200x^2z^3 + 125xz^4$

60. $15x^2y^3 - 95xy^2 - 770y$