4.3 Factoring $x^2 + bx + c$

The next polynomial that we want to factor is the trinomial. We will work on factoring trinomials in over the next two sections.

Our ultimate goal is to be able to factor a trinomial in general, that is $ax^2 + bx + c$

However, just to get a feel for how factoring trinomials works, and what they look like when they are factored, we start with a = 1.

There are numerous different processes we can use to factor a trinomial. For the sake of simplicity, we will only present one in this text. The process is called “factoring by trial factors”.

The idea in trial factors is that when a trinomial factors, it factors as two binomials. If we think back to our multiplication from chapter 3, we can see this is the case. That is to say, when we multiplied two binomials together, we almost always ended up with a trinomial.

So, since all factorable trinomials factor as two binomials, the idea for factoring is to simply “guess” at how it factors, and then re-multiply to see if we did it correctly.

However, we do have some guidelines as to our guessing. The back term in a trinomial always comes from the back terms of the binomial multiplied together. So we have the following.

<u>Factoring Trinomials by Trial Factors</u>

$$x^2 + bx + c = (x + \underline{\quad\quad})(x + \underline{\quad\quad})$$

Multiply to “c”

So again, the idea is we take an educated guess at the way it factors and then multiply the binomials out to check to see if the middle term is correct. We tend not to worry about the leading term in this case because we set up the binomials so that it works as well.

There are a couple of other things to keep in mind when doing this, but we will address these things as they come up in the examples.

<u>Example 1</u>:

Factor.

a. $x^2 + 5x + 4$ b. $y^2 - 10y + 21$ c. $x^2 - 4x + 6$

Solution:

a. The first thing we need to address is, when we are building our binomials for a factored trinomial, there are rules that we need to follow for the signs in the binomials. We will give you these rules as they come up. In this case, since all the terms are positive, in the binomial, all the terms must be positive.

   Another way of saying this is “**all positive means all positive**”.

   So, since, as stated above, if a trinomial factors, it will factor as two binomials, we begin by building our binomials. We start by plugging in the variables in a way that makes sense. Meaning here, since the front term has x^2, each front should have an x.

This gives us

$$x^2 + 5x + 4$$
$$= (x+ \quad)(x+ \quad)$$

Now, the back term in the binomials need to multiply together to produce the back of the trinomial. That is to say, we need the back terms need to multiply to get 4. So our options for this are 2 times 2, or 4 times 1.

To figure out which one is correct, we can just try one, and multiply it out to check. Let's say we try the 2 times 2. This gives

$$x^2 + 5x + 4$$
$$= (x + 2)(x + 2)$$

Now multiply like we learned in chapter 3 to check gives

$$= x^2 + 2x + 2x + 4$$
$$= x^2 + 4x + 4$$

Since this is not correct, the 2 and 2 trial is not correct. So then we need to try our other option, 4 times 1. We have

$$x^2 + 5x + 4$$
$$= (x + 4)(x + 1)$$

Multiplying to check gives

$$= x^2 + 4x + x + 4$$
$$= x^2 + 5x + 4$$

Since we got back our original trinomial, this must be the correct way to factor. So,

$$x^2 + 5x + 4$$
$$= (x + 4)(x + 1)$$

b. Again, we start with addressing the rule for the signs of a factored trinomial. In this case, whenever the back term is positive, and the middle term is negative, both binomials will have a negative. The idea behind this is, since the back term needs to be the product of the two back terms, if it is positive, it must be either two positives or two negatives. Since the middle is negative, it is logical that they need to be negative.

So, the rule is "**if the back is positive and the middle is negative, both binomials are negative**".

Now, we start setting up our binomials so that the variables make sense and we follow the rule for the signs. In this case,

$$y^2 - 10y + 21$$
$$= (y- \quad)(y- \quad)$$

Here, to fill in the trial, the back terms need to multiply to 21. Our options are 1 and 21, or 3 and 7. It turns out if we try the 1 and 21, it will not work when we multiply out to check. So, lets try the 3 and 7. We have

$$y^2 - 10y + 21$$
$$= (y - 3)(y - 7)$$

Multiplying to check gives

$$(y - 3)(y - 7)$$
$$= y^2 - 3y - 7y + 21$$
$$= y^2 - 10y + 21$$

So our factoring is correct,

$$y^2 - 10y + 21 = (x - 3)(x - 7)$$

c. Lastly, in this example, we follow the rules that we have already established. Namely, the back is positive and the middle is negative, so they are both negative. This gives,

$$x^2 - 4x + 6$$
$$= (x- \quad)(x- \quad)$$

Since the back needs to multiply to 6, our options are 1 and 6, or 2 and 3. If we try the 1 and 6 we get

$$x^2 - 4x + 6$$
$$= (x - 1)(x - 6)$$

Multiply to check

$$= x^2 - x - 6x + 6$$
$$= x^2 - 7x + 6$$

This is not correct. So we try the 2 and 3

$$x^2 - 4x + 6$$
$$= (x - 2)(x - 3)$$

Multiply to check

$$= x^2 - 2x - 3x + 6$$
$$= x^2 - 5x + 6$$

This is also not correct.

Since we are out of options, it looks like we cannot factor this trinomial.

In fact, occasionally, we run across polynomials that will not factor. When this is the case, we say that the polynomial is **prime**.

So here, we simply say $x^2 - 4x + 6$ is prime.

Now that we have the idea, let's try some harder ones.

Example 2:

Factor.

a. $2a^2 + 26a + 24$ b. $x^2y^3 - 4xy^2 - 21y$ c. $x^2 + 2xy + y^2$

Solution:

a. For this trinomial, the first thing we need to recall is that we must always start with factoring out any GCF. Clearly, the GCF here is 2. Once we factor that out we can proceed with our trial factors.

$$2a^2 + 26a + 24$$
$$= 2(a^2 + 13a + 12)$$

Now, recall that "all positive means all positive" for our binomials. Also, our options for 12 are:1 and 12, 2 and 6, 3 and 4. If we try the 2 and 6 or the 3 and 4 it turns out they do not work. But the 1 and 12 do as follows.

$$2(a^2 + 13a + 12)$$
$$= 2(a + 1)(a + 12)$$

Checking by multiplying would verify that we get $a^2 + 13a + 12$ on the inside.

b. Again, we start with the GCF.

$$x^2y^3 - 4xy^2 - 21y$$
$$= y(x^2y^2 - 4xy - 21)$$

This time, the variables in the trinomial are different then they have been so far. So, we have to start with placing them into the binomials in a way that make sense. Since the first term has both an x^2 and a y^2, it would make sense to have an x and a y in each binomials first term. This gives

$$y(x^2y^2 - 4xy - 21)$$
$$= y(xy \qquad)(xy \qquad)$$

Also, we need to deal with the signs. In this case (which is the only other case for signs) we notice that the back is negative. The rule here is "**if the back is negative, then one is positive and one is negative**".

So how do we determine which is which? As it turns out, you can put them wherever you want and if they are in the wrong spot, you will know because the sign of the middle term will be off. If that happens, you simply switch the signs in the binomials and it will work. We have

$$y(xy - \quad)(xy + \quad)$$

Let's show you. Let's factor with 3 and 7 as follows

$$y(x^2y^2 - 4xy - 21)$$
$$= y(xy - 3)(xy + 7)$$

Now if we multiply to check we get

$$y(xy - 3)(xy + 7)$$
$$= y(x^2y^2 + 7xy - 3xy - 21)$$
$$= y(x^2y^2 + 4xy - 21)$$

Notice that the sign in the middle term is off. So, if we switch the places of the positive and negatives, the factoring should work.

$$y(x^2y^2 - 4xy - 21)$$
$$= y(xy + 3)(xy - 7)$$

Now multiplying gives

$$y(xy + 3)(xy - 7)$$
$$= y(x^2y^2 - 7xy + 3xy - 21)$$
$$= y(x^2y^2 - 4xy - 21)$$

So our trinomial factors as

$$x^2y^3 - 4xy^2 - 21y = y(xy + 3)(xy - 7)$$

c. Finally, we notice that this trinomial does not have a GCF. However, the variables, again, are a little differently arranged. In this case, since the front term has x's and the back term has y's, it would make sense to make the front terms have x's and the back terms have y's. Also, since there is no coefficient on the back term, it must be a 1. Therefore, the only option we have for factoring is using a 1 and 1. We get

$$x^2 + 2xy + y^2$$
$$= (x + y)(x + y)$$

This can easily be verified by multiplying. Normally we would have been done. However, notice that we have the same binomial twice. When this happens, we should condense them down as an exponent.

$$(x + y)(x + y)$$
$$= (x + y)^2$$

One additional note about factoring, in general. Since, when factoring, we produce a product (that is, a bunch of stuff multiplied together) the order that the resulting factors shows up does not matter.

So, if you are working on a problem and you get $(x-3)(x-5)$, for example, but upon looking at the answer see a $(x-5)(x-3)$. You should be aware that they are the same answer. The order doesn't matter. Be careful though, the positives and negatives should still be with the same numbers, just the order of the binomial factors themselves doesn't matter.

4.3 Exercises

Factor.

1. $x^2 - 2x - 15$
2. $x^2 + 2x - 15$
3. $a^2 + 2a - 63$
4. $n^2 + 4n + 4$
5. $x^2 + x + 3$
6. $r^2 - 9r + 14$
7. $y^2 + 11y + 24$
8. $x^2 - 15x + 54$
9. $a^2 - 4a + 4$
10. $x^2 + 6x - 27$
11. $c^2 - 4c - 21$
12. $a^2 - 4ac - 21c^2$
13. $b^2 + 6b - 27$
14. $a^2b^2 - 11ab + 24$
15. $x^2 + 13xy + 36y^2$
16. $z^2 + 5a - 36$
17. $x^2y^2 - 4xy + 3$
18. $d^2 - 2d - 80$
19. $x^2 - 16x + 64$
20. $x^2 + 8xy + 12y^2$
21. $x^2 - 23x - 210$
22. $p^2 + 6p - 16$
23. $a^2 - 2ab - 18b^2$
24. $y^2 - 2y - 80$
25. $x^2 + x + 1$
26. $x^2 - 6x + 9$
27. $n^2m^2 - 6nm - 16$
28. $a^2 - 3a + 9$
29. $x^2 + 8xy + 15y^2$
30. $x^2y^2 + 15xy + 50$
31. $5x^2 - 35x + 60$
32. $3x^2 - 39x + 126$
33. $3a^2 + 9a - 12$
34. $6b^2 + 12b - 48$
35. $x^3 + 20x^2 + 100x$
36. $2y^2 + 8y + 8$
37. $-x^2 + 2x + 195$
38. $-x^2 + 3x + 154$
39. $-x^2y - 7xy + 44y$
40. $-2x^8 - 4x^7 + 160x^6$
41. $x^3y + 16x^2y + 64xy$
42. $2x^3 + 36x^2 + 162x$
43. $4a^2b^2 + 8ab^2 - 32b^2$
44. $2x^3y + 2x^2y - 40y$
45. $5a^2b - 65ab^2 + 210b^3$
46. $6cx^2 - 66cx + 168c$
47. $-2ax^2 - 16axy + 96ay^2$
48. $-2x^3 + 2x^2 + 110x$
49. $-3a^7 - 24a^6 + 144a^5$
50. $4x^{10} + 8x^9 - 396x^8$
51. $3x^4 - 42x^3 + 147x^2$
52. $2a^2bc + 6abc - 20bc$
53. $5x^4 + 50x^3y + 105x^2y^2$
54. $2x^3 + 16x^2 + 32x$
55. $7x^5y^2 + 98x^4y + 231x^3$
56. $9x^7 + 27x^6 - 252x^5$
57. $2a^9b^3 + 4a^8b^3 - 160a^7b^3$
58. $4a^2b - 52ab + 168b$
59. $5x^8y^3z + 10x^7y^2z - 495x^6yz$