4.2 Factoring by Grouping

Now that we can factor out a GCF, we need to turn our attention to factoring all of the other polynomials we will encounter.

We will approach factoring by basing our technique on the number of terms that a polynomial has. We will start with the larger polynomials and work our way down to the smaller polynomials.

So, we start with factoring a 4-term polynomial.

Example 1:

Factor $ax + ay + bx + by$

Solution:

> The first thing we should always do in a factoring problem is look for a GCF. Notice that this polynomial does not have any GCF that would run through the entire polynomial.
>
> However, we should notice that there is a GCF for the first two terms alone, and a GCF for the last two terms alone. Let's begin by factoring those out.
>
> That is to say, factor the a out of the first two terms, and the b out of the back two terms.
>
> $$ax + ay + bx + by$$
>
> $$= a(x + y) + b(x + y)$$
>
> Now, as we saw in the last section, we have a common binomial factor. So let's factor it out.
> $$a(x + y) + b(x + y)$$
>
> $$= (x + y)(a + b)$$
>
> This polynomial is now factored.

The process we just saw of factoring by taking a GCF out of pairs of terms is called **factoring by grouping.** It is the primary process we use for factoring polynomials which have 4 terms.

Here are the steps we can use for the grouping method.

Factoring a 4-term Polynomial by Grouping
1. Arrange the terms so that the first two have a common factor and the last two have a common factor.   2. For each pair, factor out the GCF.   3. If step 2 produces a common binomial factor, factor it out.   4. If step 2 does not produce a common binomial factor, the rearrange the terms and try again.   5. If step 4 never works, then the polynomial will not factor by grouping.

There are a couple of things that can happen while using this process that you need to be looking for. We will address these issues as they come up in the examples.

We will start with some of the simpler examples to get us going.

<u>Example 2</u>:

Factor.

    a. $2ax^2 - cx^2 + 6a - 3c$ b. $a^2x + 3a^2y - 3x - 9y$ c. $xy - 5y - 2x + 10$

Solution:
   a. First, we notice that our polynomial has 4 terms. If this is the case, we want to factor by grouping. Also, notice that the first two terms already have a common factor and the last two terms already have a common factor.

       So we can factor out the GCF from each pair as they are.

$$2ax^2 - cx^2 + 6a - 3c$$

$$= x^2(2a - c) + 3(2a - c)$$

       Notice that we have a common binomial factor. So we can factor it out as we did in the last section. This gives

$$x^2(2a - c) + 3(2a - c)$$

$$= (2a - c)(x^2 + 3)$$

   b. Again, notice that the first two terms already have a common factor and the last two have a common factor.

       However, you need to notice that the first term in the second pair (that is the 3x) is a negative term. This is important because **any time the first term in the second grouping is negative, you want to factor the negative out with the GCF**.

       So, what happens when you factor a negative out? Factoring a negative out of a group of terms does the same thing that multiplying a negative though does. That is, <u>factoring a negative out of a polynomial changes the sign of each term in the polynomial</u>.

       So, in the case here, when we factor the -3 out of the back two terms, we will end up with a x and +3. We have

$$a^2x + 3a^2y - 3x - 9y$$

$$= a^2(x + 3) - 3(x + 3)$$

       Now, we factor out the common binomial.

$$a^2(x + 3) - 3(x + 3)$$

$$(x + 3)(a^2 - 3)$$

   c. Lastly, we follow the same process as the other parts above. Each pair (grouping) already has a GCF. So we start with factoring it out and then factor out the common binomial factor.

$$xy - 5y - 2x + 10$$

Factor y out of the first group, and -2 out of the second group (which changes the signs)

$$= y(x - 5) - 2(x - 5)$$

Factor out the common x - 5

$$= (x - 5)(y - 2)$$

Now, let's take a look at some problems that are a little more difficult.

Example 3:

Factor.

   a. $4a^2 + 5ab - 10b - 8a$ b. $6ax + 3bd - 2ad - 9bx$ c. $3x^3 + 3x^2 - 6x - 6$

Solution:
   a. The first thing we should notice is that as the polynomial sits, the first two do have a common factor, and the last two also have a common factor. So, as we did in example 2, lets try factoring as it is. We get
$$4a^2 + 5ab - 10b - 8a$$

$$= a(4a + 5b) - 2(5b + 4a)$$

      The question now becomes, are those two binomials the same. In order for two polynomials to be the same, they have to have all the exact same terms, but not necessarily in the same order.

      That means, two polynomials are the same if term by term they are the same. So, even though they are written in a different order, the binomials are the same, term by term. That means we can factor it out. Also, since they are the same, it doesn't matter which one you put in the front as the GCF. We choose to use the one which has the terms in alphabetical order.
$$a(4a + 5b) - 2(5b + 4a)$$

$$= (4a + 5b)(a - 2)$$

   b. This time, we notice that even though the first two terms have a common factor, the last two terms do not have a common factor. So, in this case, it might be a good idea to reorder the terms so that each grouping has a GCF.

      The best way to do this is to put the terms together that seem like they have the most in common. So let's reorganize and factor as follows

$$6ax + 3bd - 2ad - 9bx$$

$$= 6ax - 9bx - 2ad + 3bd \qquad \text{Factor the GCF of each pair}$$

$$= 3x(2a - 3b) - d(2a - 3b) \qquad \text{Factor out the common binomial}$$

$$= (2a - 3b)(3x - d)$$

   c. Finally, the first thing that we need to notice is that the entire polynomial has a GCF of 3. The first thing that must always happen with factoring is, if there is a GCF, it must be factored out. So, we factor out the 3, then proceed with grouping as usual.

$$3x^3 + 3x^2 - 6x - 6 \qquad \text{Factor out 3}$$

$$= 3(x^3 + x^2 - 2x - 2)$$

$$= 3(x^2(x + 1) - 2(x + 1)) \qquad \text{Factor the GCF from the pairs}$$

$$\qquad\qquad\qquad\qquad\qquad\quad \text{Factor out the common x + 1}$$

$$= 3(x + 1)(x^2 - 2)$$

4.2 Exercises

Factor.

1. $ax + ay + bx + by$

2. $bx - by + cy - cx$

3. $ay + yz + ab + bz$

4. $az + at + bz + bt$

5. $y^2 + xy + ay + ax$

6. $3a - 3b + a^2 - ab$

7. $x^2 + xt + ax + at$

8. $4x - 8y - ax + 2ay$

9. $x^2 + 4y - xy - 4x$

10. $ax^2 - ay + by - bx^2$

11. $ax + bx - a - b$

12. $ax^3 + 2ay - x^3 - 2y$

13. $x^2 - ax - xy + ay$

14. $a^2 - 4b - 4a + ab$

15. $2c + 2d - cd - d^2$

16. $3c - cd + 3d - c^2$

17. $x^2 - xy - 4x + 4y$

18. $ab - bc - ad + bd$

19. $7x + y^2 - 7y - xy$

20. $5x^3 + 15x + 3x^3 + 9$

21. $at^2 - 3x + 3ax - t^2$

22. $a^3 - 8a^2 + 6a - 48$

23. $2x^2 + x^4 + 4y + 2x^2y$

24. $2b^2 - cb - 2b + c^2$

25. $m^2 + 3m - 3n - nm$

26. $4x^3 + 5x^2 + 20x + 25$

27. $x^4 - 2x^3 + 3x - 6$

28. $b^2c^2 - 3bc^2 - ab + 3a$

29. $2a^3 - 2a^2 - 2ax + 2x$

30. $2x^2 + 8x + 10xy + 40y$

31. $2ax^2 + 2ay^2 - 2bx^2 - 2by^2$

32. $x^3y + 9x^2y + 4xy + 36y$

33. $x^3 - ax^2 - 2x^2 + 2ax$

34. $x^3y^2 + 6x^2y^2 + 8xy^2 + 48y^2$

35. $2x^4 - 8x^3 + 10x^2 - 40x$

36. $14a^3b + 28ab + 4a^2b + 8b$

37. $3x^5y + 9x^4y + 9x^3y + 27x^2y$

38. $5x^5 - 10x^4 + 25x^3 - 50x^2$

39. $x^2y^2z^2 + xy^2zt + xy^3z + y^3t$

40. $6xy^3z + 36xy^2z + 36xyz + 216xz$