3.5 Dividing Polynomials

The final basic operation that we want to look at is division of polynomials.

Dividing polynomials can be challenging, however, we will see, it does have a process. If we can simply learn the process, division isn't that difficult.

We will take a look at two types of division.

We start with

Dividing by a Monomial

The best way to deal with division by a monomial is by seeing an example. It will basically come down to splitting up the division, and knowing your properties of exponents.

Example 1:

Divide

a. $\frac{6x^7-9x^5+15x^4-3x^3}{3x^3}$

b. $(30x^6 - 30x^5 + 25x) \div 5x$

Solution:

a. To divide a polynomial by a monomial, all we need to do is split the division up over the individual terms, then reduce each resulting term by using properties of exponents. We get

$$\frac{6x^7 - 9x^5 + 15x^4 - 3x^3}{3x^3}$$

$$= \frac{6x^7}{3x^3} - \frac{9x^5}{3x^3} + \frac{15x^4}{3x^3} - \frac{3x^3}{3x^3}$$

Reduce coefficients and Subtract powers

$$= 2x^4 - 3x^2 + 5x - 1$$

b. First we need to write it as a fraction. Then, like part a above, we split the polynomial up, then reduce.

$$(30x^6 - 30x^5 + 25x) \div 5x$$

$$= \frac{30x^6 - 30x^5 + 25x}{5x}$$

$$= \frac{30x^6}{5x} - \frac{30x^5}{5x} + \frac{25x}{5x}$$

$$= 6x^5 - 6x^4 + 5$$

Division by a monomial is fairly simple. But what do we do if we are dividing by a polynomial with more than one term. In this case, we have to use long division.

The process of long dividing polynomials is just like the process of long dividing numbers. However, we do need to be aware of a few things when doing this process.

Here are some things to keep in mind as we get started on long division of polynomials.

Long Division of Polynomials
Key ideas: 1. Always match the leading terms exactly 2. Be careful with the subtraction step 3. Always add the remainder piece as a new term 4. Make sure all powers of the variable are present

With these key ideas in mind, let's look at some division.

Example 2:

Divide

a. $\frac{x^2+7x+10}{x+5}$ b. $(x^2+6x+10)\div(x+3)$ c. $\frac{6x^4-x^3-2x^2-7x-19}{2x-3}$

Solution:

a. The first thing we need to do is express the problem as long division. We get

$$x+5\overline{\smash{\big)}x^2+7x+10}$$

Now we need to follow our key ideas above. The first key idea is that we want to "match the leading terms exactly". This means, we want the leading term in the divisor (x) to look exactly the same as the leading term in the dividend (x^2). To do this we would multiply by x.

So, we place the x on top of the division bar, and (as we would do with standard division) multiply the x on top by the x+5 out front. This gives us $x^2 + 5x$. We place that under the division bar as below.

$$\begin{array}{r} x \\ x+5\overline{\smash{\big)}x^2+7x+10} \\ \underline{x^2+5x} \end{array}$$

Now we are at another critical step. Here, we subtract. We have to be very careful, though, because we are subtracting the entire bottom row. This means, we have to distribute a negative and change all of the signs (since subtracting $x^2 + 5x$ means $-(x^2 + 5x)$. So we have

$$\begin{array}{r} x \\ x+5\overline{\smash{\big)}x^2+7x+10} \\ \underline{-x^2-5x} \\ 2x \end{array}$$

Just like standard division, now we bring down the next term and start over by matching the leading term.

$$
\begin{array}{r}
x+2 \\
x+5\overline{)\,x^2+7x+10} \\
\underline{-x^2-5x} \\
2x+10
\end{array}
$$

So, to make x look like 2x we multiply by 2. We get 2(x + 5) = 2x + 10. Placing it below we have

$$
\begin{array}{r}
x+2 \\
x+5\overline{)\,x^2+7x+10} \\
\underline{-x^2-5x} \\
2x+10 \\
\underline{2x+10}
\end{array}
$$

Now, again, subtract the bottom by distributing a negative. This gives us

$$
\begin{array}{r}
x+2 \\
x+5\overline{)\,x^2+7x+10} \\
\underline{-x^2-5x} \\
2x+10 \\
\underline{-2x-10} \\
0
\end{array}
$$

So our answer is x + 2.

b. Again, we start by writing the problem as long division, and we proceed as we did in part a.

$x+3$	Make x look like x^2
$x+3\overline{)\,x^2+6x+10}$	Multiply and distribute the subtraction
$\underline{-x^2-3x}\ \searrow$	
$3x+10$	Make x look like 3x
$\underline{-3x-9}$	Multiply and distribute the subtraction
1	

Here, we have a remainder. So according to our “key ideas,” we need to add the remainder piece on. The way we do this is by placing the remainder over the divisor (the number in front of the division bar) and attaching it as another term of the solution.

This gives us a solution of $x+3+\frac{1}{x+3}$.

c. Finally, we divide as we did in all of the other parts of this example. Start by putting it into long division. Then match the leading terms and carefully distribute the subtraction on every step.

For the sake of space, we will put the entire process on the next page.

$$
\begin{array}{r}
3x^3 + 4x^2 + 5x + 4 \\
2x-3\overline{\big)\,6x^4 - x^3 - 2x^2 - 7x - 19} \\
\underline{-6x^4 + 9x^3} \qquad\qquad\qquad\quad \\
8x^3 - 2x^2 \qquad\qquad\quad \\
\underline{-8x^3 + 12x^2} \qquad\qquad\quad \\
10x^2 - 7x \qquad\quad \\
\underline{-10x^2 + 15x} \qquad \\
8x - 19 \\
\underline{-8x + 12} \\
-7
\end{array}
$$

Match leading terms and distribute the subtraction on each step

Bring down the next terms

So, attaching the remainder part, our answer is $3x^3 + 4x^2 + 5x + 4 - \frac{7}{2x-3}$

Now that we have the basic idea, let's look at some more difficult division problems.

Example 3:

Divide

a. $(x^4 - 14) \div (x - 2)$

b. $\frac{x^4 - 2x^3 - x + 2}{x^2 - x - 1}$

Solution:

a. To start we notice that we are missing some of the powers of x. Meaning we are missing the x^3, x^2, and x. So, to account for this, we insert place holders for each of the missing terms. The place holders are simply there to provide some stability the division and they make the entire process much easier to deal with.

The place holders are simply the missing term, with a zero coefficient. That is, we add in the terms $0x^3$, $0x^2$, and 0x. We then long divide as usual. As always, be careful with the subtraction. Match the leading terms, then distribute and change the signs in the subtracting step.

We have

$$
\begin{array}{r}
x^3 + 2x^2 + 4x + 8 \\
x-2\overline{\big)\,x^4 + 0x^3 + 0x^2 + 0x - 14} \\
\underline{-x^4 + 2x^3} \qquad\qquad\qquad\quad \\
2x^3 + 0x^2 \qquad\qquad\quad \\
\underline{-2x^3 + 4x^2} \qquad\qquad\quad \\
4x^2 + 0x \qquad\quad \\
\underline{-4x^2 + 8x} \qquad\quad \\
8x - 14 \\
\underline{-8x + 16} \\
2
\end{array}
$$

So we have $x^3 + 2x^2 + 4x + 8 + \frac{2}{x-2}$

b. Lastly, we need to apply the same concept to dividing by larger polynomials. The good news is, the process doesn't change at all. We simply long divide like we did for all of the other examples. It's just a little more challenging with a longer polynomial.

Again, always be careful of matching the leading terms and distributing the negative when you doing the subtraction step.

Also, notice we are missing a x^2 term, so we need a place holder of $0x^2$ in its spot. We get

$$\begin{array}{r} x^2 - x \\ x^2 - x - 1 \overline{\big) x^4 - 2x^3 + 0x^2 - x + 2} \\ \underline{-x^4 + x^3 + x^2} \\ -x^3 + x^2 - x \\ \underline{x^3 - x^2 - x} \\ -2x + 2 \end{array}$$

Notice, at this point, we can no longer match the leading terms. This means we must be done with our division. As it turns out, as soon as the degree on the bottom (right after subtraction) is smaller than the degree of the divisor, the division is complete. We just bring down the rest of the terms and attach the remainder.

So we have an answer of $x^2 - x + \frac{-2x+2}{x^2-x-1}$

3.5 Exercises

Divide.

1. $\frac{6x^3-15x^2+18}{3}$
2. $\frac{8x^3-24x+18}{2}$
3. $(10x^7 - 20x^3 + 25x) \div 5x$
4. $(21x^7 - 6x^5 + 15x^4 - 3x^2) \div 3x^2$
5. $\frac{27x^{10}-21x^9-12x^7-3x^4}{-3x^3}$
6. $\frac{32x^9-56x^8}{-8x^5}$
7. $(56x^9 - 24x^7) \div (-8x^4)$
8. $\frac{16x^{10}-12x^8-6x^6-2x^2}{-2x^2}$
9. $\frac{10x^9y^{10}-14x^7y^8+12x^3y^4}{2x^3y}$
10. $\frac{36x^8y^9-8x^6y^7+20x^2y^3}{4x^2y}$
11. $\frac{x^2+9x+20}{x+4}$
12. $\frac{x^2-18x+81}{x-9}$
13. $(x^2 - 13x + 36) \div (x - 9)$
14. $\frac{6x^2-4x+5}{x-1}$
15. $\frac{x^2+5x-30}{x+8}$
16. $\frac{3x^2-10x+1}{x-4}$
17. $\frac{12x^2-30x+36}{3x+3}$
18. $(x^2 + 4x + 4) \div (x + 2)$
19. $\frac{x^3-x^2+x-6}{x+1}$
20. $\frac{6x^2-20x+125}{2x+8}$

21. $(x^3 - x^2 - x - 3) \div (x + 1)$

22. $\frac{36x^3-64x^2-88x-16}{9x+2}$

23. $\frac{2x^3+3x^2+5x-8}{x+2}$

24. $\frac{x^3+4x^2+x-24}{x+7}$

25. $\frac{4x^3-8x^2+7x-2}{2x-1}$

26. $\frac{x^4-x^3-x^2-x+1}{x-2}$

27. $\frac{3x^3-7x^2-22x+8}{3x-1}$

28. $\frac{36x^3-64x^2-88x-16}{9x+2}$

29. $\frac{21x^3-29x^2-24x-4}{7x+2}$

30. $\frac{3x^3-4x^2+x-17}{x-4}$

31. $(x^2 - 16) \div (x + 4)$

32. $\frac{x^3+64}{x+4}$

33. $(x^3 - x + 6) \div (x - 3)$

34. $\frac{x^3+7x-44}{x+9}$

35. $\frac{4x^3-29x-38}{2x+3}$

36. $\frac{x^3-x+30}{x-6}$

37. $\frac{x^4+2x^2-60x+26}{x-4}$

38. $\frac{27x^3-39x-26}{3x+2}$

39. $\frac{x^4+15x^2-19x+93}{x-2}$

40. $(x^5 + 2) \div (x + 3)$

41. $\frac{5x^4-3x^2+2}{x^2-3x+5}$

42. $\frac{4x^4-x^3+2x-1}{2x^2-3x-4}$

43. $\frac{3x^3-4x^2-3}{x^2+5x+2}$

44. $\frac{x^3+x^2+x+1}{x^2+x+1}$

45. $\frac{x^3+6x+3}{x^2-2x+2}$

46. $\frac{4x^3-x^2+8x-1}{x^2-x+1}$

47. $\frac{x^4-4x^3+4x^2-16}{x^2-2x+4}$

48. $\frac{3x^4-x^2+1}{x^2-x+2}$

49. $(3x^4 - 2x^3 + 2) \div (x^2 - 1)$

50. $(2x^5 - 2x - 1) \div (x^2 - 2)$