3.4 Multiplying Polynomials

Let's turn our attention to the next basic operation on polynomials, multiplication.

There are a number of ways to learn how to multiply polynomials, however, they all boil down to the same one rule. Any other method (of which we will address a couple) are all special cases of the same one rule.

The rule is

Multiplying Two Polynomials
Multiply each term in the first polynomial by each term in the second polynomial, and then combine like terms.

We can say this also very simply as everything in the first, times everything in the second polynomial, or "**everything times everything**".

Let's see some examples.

Example 1:

Multiply.

a. $6x(4y+7z)$ b. $-ab(a^4-a^2b^2-b^4)$ c. $(x+5)(x-3)$

d. $(x-2y)(2x^2-3xy+y^2)$ e. $(2x+1)(x+2)(2x+2)$

Solution:

a. According to our rule, we simply need to multiply each term in the first polynomial, that is, the monomial, times each term in the second polynomial, that is the binomial.
So we have

$$6x(4y+7z)$$
$$=24xy+42xz$$

It looks the same as the distributive property that we learned in chapter 1, however, keep in mind that the distributive property has severe limitation when it comes to polynomials. That is why we want to always think of the rule "everything times everything".

b. Again, we multiply the monomial times each term in the trinomial. We must make sure that we follow our properties of exponents (that is, add the exponents) as we do this.
We get

$$-ab(a^4-a^2b^2-b^4)$$
$$=-a^5b+a^3b^3+ab^5$$

c. Here, we will really see the "everything times everything" rule start to kick in. We have to make sure each of the terms in the first binomial gets multiplied by each of the terms in the second binomial.

This means, we have to multiply the first term in the x + 5 (so the x), by each term in the x - 3, then the second term in the x + 5 (so the 5), by each term in the x - 3.

It looks like the following

$$(x+5)(x-3)$$

$$= x^2 - 3x + 5x - 15$$

x in the first times each in the second

5 in the first times each in the second

Now we just need to combine like terms and we are done. We get

$$\begin{aligned} &x^2 - 3x + 5x - 15 \\ &= x^2 + 2x - 15 \end{aligned}$$

d. In this case, the multiplication is done exactly the same, we just have more terms to deal with. We still start by multiplying each term in the first polynomial by each term in the second polynomial. Then combine like terms. We get

$$(x-2y)(2x^2-3xy+y^2)$$

$$= \underbrace{2x^3 - 3x^2y + xy^2}_{\text{x times each in the second}} \underbrace{- 4x^2y + 6xy^2 - 2y^3}_{\text{-2y times each in the second}}$$

$$= 2x^3 - 7x^2y + 7xy^2 - 2y^3$$

e. Lastly, since the rule only works for multiplying two polynomials at a time, we will need to multiply the first two polynomials together first, and then multiply that answer by the last polynomial, making sure to follow the “everything times everything” rule the entire time.

We get

$$(2x+1)(x+2)(2x+2)$$

$$= (2x^2+4x+x+2)(2x+2)$$ Combine like terms

$$= (2x^2+5x+2)(2x+2)$$ Everything times everything again

$$= 4x^3 + 4x^2 + 10x^2 + 10x + 4x + 4$$ Combine like terms

$$= 4x^3 + 14x^2 + 14x + 4$$

Multiplying doesn’t get much more complicated than this. The most difficult part is making sure that each term gets multiplied by each term, and not leaving anything out. If we are careful, multiplication is quite simple.

As it turns out, there is a very popular rule for multiplying binomials together. We use the alliteration FOIL. Here is how it works.

The Binomial Rule **(or FOIL Method)**
F-irst: Multiply the first terms of the two binomials O-utside: Multiply the outside terms of the two binomials I-nside: Multiply the inside terms of the two binomials L-ast: Multiply the last terms of the two binomials

Here is an example of just how this works.

Example 2:

Multiply.

$(3x + 2)(x - 4)$

Solution:

The way FOIL works is you simply follow the alliteration. So we multiply the first terms in each binomial together (the 3x and the x) then the outside terms (the 3x and the -4) , the inside terms (the 2 and the x), then lastly the last terms (the 2 and the -4).
This gives us

$$(3x + 2)(x - 4)$$

$$= \underset{\text{First}}{3x^2} \quad \underset{\text{Outside}}{-12x} \quad \underset{\text{Inside}}{+2x} \quad \underset{\text{Last}}{-8}$$

$$= 3x^2 - 10x - 8$$

The problem with the FOIL method is that is **ONLY** works on binomials. Also, since we are clearly still multiplying the same terms as we would have with the primary “everything times everything” method, we always get the same answer.

With this being said, since the “everything time everything” rule works every time, it is far easier to just remember this one rule and apply it to every situation, instead of remembering different rules for every different situation.

For this reason, we will rely mostly upon the “everything times everything” rule.

Now, multiplying contains some quirks. Let’s look at a few.

Example 3:

Multiply.

a. $(3x + 2)(3x - 2)$ b. $(x + 4)^2$ c. $(2x - 1)^2$

Solution:

a. As we stated, since the “everything times everything” rule always works, let’s use it to multiply these binomials together. So we multiply the 3x times each term in the second binomial, then -2 times each term in the second binomial.
We get

$$(3x + 2)(3x - 2)$$

$$= 9x^2 - 6x + 6x - 4$$

$$= 9x^2 - 4$$

So, in this case, we multiplied two binomials together and got a seemingly very special looking binomial out of it.

b. In this case, we must be very careful. We know from our properties of exponents (from section 3.1) that we can "pull through" exponents from the outside. However, we can only do so when we have ONLY multiplying or dividing on the inside.

In this case, we have adding on the inside. So, instead, we should do what an exponent tells us to do… multiply the binomial times itself. That is, we should write it out twice.

$$(x+4)^2$$

$$= (x+4)(x+4)$$

Now multiply as usual.

$$(x+4)(x+4)$$

$$= x^2 + 4x + 4x + 16$$

$$= x^2 + 8x + 16$$

c. Just like in part b, above, we need to start with writing out the binomial twice and multiply as usual.

$$(2x-1)^2$$

$$= (2x-1)(2x-1)$$

$$= 4x^2 - 2x - 2x + 1$$

$$= 4x^2 - 4x + 1$$

Caution: It is extremely common to make the error of pulling a power through a binomial. **ALWAYS** write it out as many times as the power indicates and multiply as usual. It is a horrific error to simply bring the exponent into the polynomial when you have adding or subtracting involved.

From the previous example, we see some special cases which appear with multiplication. We get the following three formulas.

Special Products
$(a+b)(a-b) = a^2 - b^2$ $(a+b)^2 = a^2 + 2ab + b^2$ $(a-b)^2 = a^2 - 2ab + b^2$

Even though we could use these formulas to multiply the special cases out, it is best to just continue using the "everything times everything" rule, since it always works.

We give you these "special product" formulas because we will need one of them in the next chapter. Its best, then, to be concerned with memorizing and using the formulas when we get there.

In the meantime, let's look at a few more challenging multiplying problems to close this section.

Example 4:

Multiply.

a. $(2A+B)^3$ b. $(x^2+2x-1)(x^2+4x-5)$ c. $(x^4-x^3-x^2-x)^2$

Solution:

a. Keeping in mind example 3, we need to start by writing this binomial out 3 times. We will then multiply as usual, “everything times everything”

$(2A+B)^3$

$=(2A+B)(2A+B)(2A+B)$ Write out 3 times
Multiply the first two polynomials

$=(4A^2+2AB+2AB+B^2)(2A+B)$ Combine like terms

$=(4A^2+4AB+B^2)(2A+B)$ Multiply the two polynomials

$=8A^3+4A^2B+8A^2B+4AB^2+2AB^2+B^3$ Combine like terms

$=8A^3+12A^2B+6AB^2+B^3$

b. Here, we simply need to apply our “everything times everything” rule to multiply. Then we will combine like terms.

$(x^2+2x-1)(x^2+4x-5)$

$=x^4+4x^3-5x^2+2x^3+8x^2-10x-x^2-4x+5$

$=x^4+6x^3+2x^2-14x+5$

c. Lastly, we will have to write this polynomial out twice, multiply and combine like terms.

$(x^4-x^3-x^2-x)^2$

$=(x^4-x^3-x^2-x)(x^4-x^3-x^2-x)$

$=x^8-x^7-x^6-x^5-x^7+x^6+x^5+x^4-x^6+x^5+x^4+x^3-x^5+x^4+x^3+x^2$

$=x^8-2x^7-x^6+3x^4+2x^3+x^2$

3.4 Exercises

Multiply

1. $3x(x^2-2x+1)$
2. $2x(4x^2+x+3)$
3. $y^2(4y^2+2y-3)$
4. $-x^2(x^2+5x-8)$
5. $-5x^2(3x^2+7x-2y)$
6. $2y^3(x^2y-2xy+10)$
7. $(x+3)(x+4)$
8. $(x-2)(x+4)$
9. $(x+10)(x-10)$
10. $(x-1)(3x-4)$
11. $(2x+1)(3x-10)$
12. $(2x-3)(2x+3)$

13. $(3x-5)(2x+1)$

14. $(2x-5)(7x+4)$

15. $(x+2y)(x-2y)$

16. $(8x-3)(3x+4)$

17. $(x-1)^2$

18. $(x+1)^2$

19. $(x+h)^2$

20. $(x-2)^2$

21. $(2x-5y)^2$

22. $(3x-1)^2$

23. $(2x+3)^2$

24. $(5x+3)^2$

25. $(5xy-2)^2$

26. $(7x+1)^2$

27. $(4x^3-3)^2$

28. $(4y^2-11)^2$

29. $(x+y)(x-y)(x^2+y^2)$

30. $(2x+y)(x-2y)(x+y)$

31. $(5x+3)(-6x^2+15x-4)$

32. $(2x-3)(x^2+5x-7)$

33. $(x^2+9)(x^2-x-4)$

34. $(3x+1)(2x^2-5x+4)$

35. $(x-2)(x^2+2x+4)$

36. $(x-3)(x^2+3x+9)$

37. $(x^2-x+1)(x^2+x+1)$

38. $(x^2-2x-1)(x^2+2x+1)$

39. $(x^2+3x-2)(x^2-3x-2)$

40. $(2x^2-x-3)(4x^2+x-1)$

41. $(x+y)^3$

42. $(x+h)^3$

43. $(x-2)^3$

44. $(x-7)^3$

45. $(3x-2y)^3$

46. $(4x+5y)^3$

47. $(x-1)(x+1)$

48. $(x-y)(x+y)$

49. $(x-1)(x^2+x+1)$

50. $(x-y)(x^2+xy+y^2)$

51. $(x-1)(x^3+x^2+x+1)$

52. $(x-y)(x^3+x^2y+xy^2+y^3)$

53. $(x-1)(x^4+x^3+x^2+x+1)$

54. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

55. $(x^2+xy+y^2+1)^2$