3.1 Exponents

We will begin this chapter with a quick refresher of what an exponent is.

Recall:

$$\underbrace{x \cdot x \cdot x \cdot x \cdot \cdots x}_{\text{n-times}} = x^n$$

n-times base exponent

So, an exponent is how we represent repeated multiplication. We want to take a closer look at the exponent.

We will begin with what the properties are for the exponents. In the following table, we will give the property, and the a simple example to illustrate the property.

Properties of Exponents	
Property	Example
1. $a^n \cdot a^m = a^{n+m}$;	$2^2 \cdot 2^3 = 2^{2+3} = 2^5$
2. $(a^n)^m = a^{n \cdot m}$;	$(x^2)^3 = x^{2 \cdot 3} = x^6$
3. $(a^n \cdot b^m)^p = a^{n \cdot p} \cdot b^{m \cdot p}$;	$(2^2 \cdot 3)^3 = 2^{2 \cdot 3} \cdot 3^{1 \cdot 3} = 2^6 \cdot 3^3$
4. $\frac{a^n}{a^m} = a^{n-m}$;	$\frac{3^4}{3^2} = 3^{4-2} = 3^2$
5. $\left(\frac{a^n}{b^m}\right)^p = \frac{a^{n \cdot p}}{b^{m \cdot p}}$;	$\left(\frac{2}{3^2}\right)^4 = \frac{2^{1 \cdot 4}}{3^{2 \cdot 4}} = \frac{2^4}{3^8}$
6. $a^{-n} = \frac{1}{a^n}$; $\frac{1}{a^{-n}} = a^n$;	$5^{-1} = \frac{1}{5}$
7. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$;	$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$
8. $a^0 = 1$;	$1{,}000{,}000^0 = 1$

Many of these properties are clear if you simply write out the exponents and the properties will become clear.

We use these properties to simplify expressions which contain exponents. What that means is, we want an expression containing exponents to have no variables or number of the same base, no negative exponents and no exponents which can be worked out.

Basically, if something can be done…. It must be done.

One other thing to always remember when working with these properties of exponents, the properties are from exponents to other exponents. So the operations we do, need to be done to the exponents of each number or variable.

Let’s start with some simple examples.

Example 1:

Simplify.

a. $(x^2y)^2$ b. $(4x^2y^3)(5xy^4)$ c. $x^{-2}y^4$ d. $\frac{a^5}{a^7}$

Solution:

a. In this example, we don't really have many things we can do. It looks like all we can do is use property 3 to bring the exponent from the outside of the parenthesis to the inside. Also, remember that even though there is no exponent of the y, everything has an invisible exponent of 1.
Putting this together we get

$$(x^2y)^2$$
$$= x^{2\cdot2}y^{1\cdot2}$$
$$= x^4y^2$$

b. This time, we have to notice that we have two expressions multiplied together. So, we need to use property 1 to get together the variables of the same base. Be careful here, even though we are multiplying them together, we are supposed to add the exponents. This gives us

$$(4x^2y^3)(5xy^4)$$
$$= 4\cdot5\cdot x^{2+1}y^{3+4}$$
$$= 20x^3y^7$$

c. Here, the only thing that needs to be addressed is that we have a negative exponent. We are not allowed to have negative exponents. So we use property 6 to make the exponent positive.

The idea behind property 6 is that whenever you have a negative exponent, you simply need to move the value across the fraction bar to make it a positive exponent. This idea works either way. This means, if something on the denominator has a negative exponent, you move it to the numerator. If something on the numerator has a negative exponent, you move it to the denominator.

In this case, we need to move the x^{-2} to the denominator (which is currently a 1) to get the sign to change. We have

$$x^{-2}y^4$$
$$= \frac{x^{-2}y^4}{1}$$
$$= \frac{y^4}{x^2}$$

d. Lastly, we just need to use property 4 to simplify. Property 4 can cause some potential issues. The idea is, you need to subtract the exponents and you have to keep in mind two things: its always "top exponent - bottom exponent" and the answer always lands on top.

If it happens to be that you get a negative exponent after that, then you deal with it at that point.

With this in mind we have

$$\frac{a^5}{a^7}$$

$$= a^{5-7}$$

$$= a^{-2}$$

Now move it to the bottom to get rid of the negative exponent.

$$= \frac{1}{a^2}$$

Now let's look at a few more difficult examples involving exponents.

<u>Example 2</u>:

Simplify.

a. $(3^{-1}a^{-2}b^3)^{-3}$ b. $\frac{4^{-1}x^{-2}y^3}{2^{-3}xy^{-5}}$ c. $\left(\frac{2^{-2}x^3}{y^{-2}}\right)^{-3}$

Solution:

a. The first thing you have to keep in mind is that there are several different ways you can work these problems. All of the different directions you could go, will still end up at the same spot, as long as you use the properties correctly.

In this case, it seems easier to start with brining the exponent through from the outside, and then proceeding from there. Remember, when using property 3 to bring the power in, you need to multiply it by each power that is already on the inside of the parenthesis.

$(3^{-1}a^{-2}b^3)^{-3}$	Multiply the power in
$= 3^3a^6b^{-9}$	Move the negative exponent to the bottom
$= \frac{3^3a^6}{b^9}$	Simplify $3^3 = 27$
$= \frac{27a^6}{b^9}$	

b. This time, it seems best to move all of our negative exponents around to make them positive before we even think about doing anything else.

Remember, we only want to move the items that have negative exponents. If something does not have a negative exponent, that is, the exponent is already positive, we need to leave it where it is.

Once we have done that, then we can decide what to do next.

$$\frac{4^{-1}x^{-2}y^3}{2^{-3}xy^{-5}}$$

$$= \frac{2^3 y^3 y^5}{4^1 x x^2}$$ Simplify 2^3 and 4^1 and combine like bases by adding exponents

$$= \frac{8y^8}{4x^3}$$ Reduce the 4 and 8

$$= \frac{2y^8}{x^3}$$

c. Similar to part a above, let's start by pulling through the exponent that is on the outside by multiplying it by each exponent on the inside.

$$\left(\frac{2^{-2}x^3}{y^{-2}}\right)^{-3}$$ Multiply through the -3

$$= \frac{2^4 x^{-9}}{y^6}$$ Move the negative exponent down and simplify 2^4

$$= \frac{16}{x^9 y^6}$$

Finally, we will take a look at some very challenging problems with exponents.

Example 3:

Simplify.

a. $\left[\left(\frac{8^{-1}x^{-2}y^2}{2^{-3}yz^{-3}}\right)^4\right]^{-1}$ b. $\frac{(3xy^2)^{-2}}{(6x^2y^{-1})^{-3}}$ c. $\frac{64x^{-4}y^{-5}}{(4^{-1}x^2y^{-1})^{-3}}$

Solution:

a. In this problem, we have many different ways we can start. We can try to get rid of all the negative exponents, we can pull though the exponents from the outside, we can get same bases together, etc.

However, in this case, it is best to start with multiplying the 4 and -1 exponents from the outside together. It doesn't seem like we are allowed to do this, but it actually is a simple usage of property 2.

Then we continue as usual

$$\left[\left(\frac{8^{-1}x^{-2}y^2}{2^{-3}yz^{-3}}\right)^4\right]^{-1}$$ Multiply 4 and -1

$$= \left(\frac{8^{-1}x^{-2}y^2}{2^{-3}yz^{-3}}\right)^{-4}$$ Bring through the -4

$$= \frac{8^4 x^8 y^{-8}}{2^{12} y^{-4} z^{12}}$$ Move the negative exponents

$$= \frac{8^4 x^8 y^4}{2^{12} y^8 z^{12}}$$ Combine the y bases by subtraction and simplify 8^4 and 2^{12}

$$= \frac{4096x^8y^{-4}}{4096z^{12}}$$

$$= \frac{x^8}{y^4z^{12}}$$

b. Here we will start by moving the "larger" negative exponents to make them positive then move along like we usually do. It looks like

$$\frac{(3xy^2)^{-2}}{(6x^2y^{-1})^{-3}}$$ Move large negative exponents

$$= \frac{(6x^2y^{-1})^3}{(3xy^2)^2}$$ Bring through the powers from the outside

$$= \frac{6^3x^4y^{-3}}{3^2x^2y^4}$$ Move the negative exponent and simplify 6^3 and 3^2

$$= \frac{216x^4}{9x^2y^4y^3}$$ Reduce and combine same bases

$$= \frac{24x^2}{y^7}$$

c. Lastly, we will work this example as we did all of the others.

$$\frac{64x^{-4}y^{-5}}{(4^{-1}x^2y^{-1})^{-3}}$$ Pull through the -3 on bottom

$$= \frac{64x^{-4}y^{-5}}{4^3x^{-6}y^3}$$ Move negative exponents

$$= \frac{64x^6}{64x^4y^3y^5}$$ Reduce and combine same bases

$$= \frac{x^2}{y^8}$$

3.1 Exercises

Simplify.

1. $x^2 \cdot x^4$
2. $y^3 \cdot y^5$
3. $(a^2)^5$
4. $(b^7)^3$
5. $\frac{x^4}{x^2}$
6. $\frac{a^{10}}{a^4}$
7. $\left(\frac{b^3}{c^4}\right)^4$
8. $\left(\frac{x^2}{y^3}\right)^3$
9. p^0

10. 34^0

11. $(9x^4)^2$

12. $(b^3c)^3$

13. $\frac{5}{x^{-1}}$

14. xy^{-2}

15. $ab^{-2}cd^0$

16. $\frac{x}{2y^{-2}z^0}$

17. $(ab^2)^{-1}$

18. $(x^{-2}y^2)^{-2}$

19. $\left(\frac{3x^{-2}}{2x^5}\right)^2$

20. $(3xy^{-1})^{-2}$

21. $(2a^3b^3)(3a^6b^2)$

22. $(3x^5y^3)(3x^6y^7)$

23. $\left(\frac{x^2}{yz^{-3}}\right)^3$

24. $\frac{xy^{-1}}{3z^{-2}}$

25. $\left(\frac{ab^4}{2a^{-2}}\right)^{-2}$

26. $\left(\frac{x^3x^{-2}}{x^5}\right)^{-2}$

27. $\left(\frac{4x^{-2}y^{-2}}{x^4x^{-1}}\right)^3$

28. $\frac{2^3x^{-3}}{4^2x^{-3}}$

29. $\left(\frac{2^{-2}x^3}{y^{-2}}\right)^{-3}$

30. $(5x^{-3}y^2)^{-3}$

31. $(27^{-1}a^{-2}b^3)^{-2}$

32. $(2x^{-3}y^{-4})^5$

33. $\left(\frac{abc}{a^{-2}b^4c^2}\right)^{-2}$

34. $\left(\frac{x^{-1}yz^{-2}}{x^{-2}y^{-2}z}\right)^{-3}$

35. $\left(\frac{24a^{-3}b^2}{36a^{-1}b^4}\right)^{-1}$

36. $\left(\frac{18x^3y^{-2}}{24x^{-1}y}\right)^{-2}$

37. $\left[\left(\frac{3m^6n^3}{4n^{-2}}\right)^2\right]^{-1}$

38. $\left[\left(\frac{x^2y^3}{2x^{-2}}\right)^{-2}\right]^{-1}$

39. $\left[\frac{6x^{-4}y^4}{(z^{-1})^{-2}}\right]^{-2}$

40. $\left[\frac{x^{-2}z^4}{2(xz)^{-2}}\right]^2$

41. $\left(\frac{a^{-2}b^{-2}c^2}{bc}\right)^{-2}$

42. $\left(\frac{7x^{-2}y^2}{21xy^{-2}}\right)^{-1}$

43. $\left(\frac{3x^{-2}y}{12x^4y^{-2}}\right)^{-2}$

44. $\left(\frac{4a^{-3}b^3}{28a^5b^{-6}}\right)^{-2}$

45. $\left(\frac{x^2y^{-1}}{3x^{-3}y^2}\right)^2$

46. $(3a^{-2}b^3)(2a^2b^{-3})^{-1}$

47. $(4a^{-3}b^4)^{-2}(2ab^{-3})$

48. $\left(\frac{4a^3b^{-3}}{24a^{-1}c^{-3}}\right)^{-2}$

49. $\frac{(3xy^2)^{-2}}{(6x^2y^{-1})^{-3}}$

50. $\frac{36x^{-4}y^{-5}}{(6x^2y^3)^2}$

51. $(2^{-3}x^{-1}y^3)(4x^{-3}y^{-2})^{-2}$

52. $(3^2a^{10}b^{-2})(3a^{-3}b^2)^{-3}$

53. $\left(\frac{3x^2y}{z^3}\right)^{-1}\left(\frac{6x^2z}{y^{-3}}\right)^2$

54. $\left(\frac{2ab^{-2}}{a^{-2}b}\right)\left(\frac{2a^{-2}b}{a^2b^3}\right)^{-2}$