2.4 Linear Inequalities and Interval Notation

We want to solve equations that have an inequality symbol instead of an equal sign.

There are four inequality symbols that we will look at: Less than <, Greater than >, Less than or equal to $\leq$ and Greater than or equal to $\geq$.

The “or equal to” types of inequalities are like combining the inequality symbol and an equal sign into the same symbol.

As usual, we are interested in finding the solutions to these inequalities.

As it turns out, these inequalities have an infinite number of solutions. So, for this reason, we want to graph these inequalities because it’s the only way to see all of the solutions at one time.

We need to start with the following facts.

<u>Graphing Inequalities</u>

- When graphing an inequality of type $\leq$ or $\geq$ we use [and] to represent the value we are working with is actually included in the solutions (hence the “or equal to part).

- When graphing an inequality of type < or > we use (and) to represent the value we are working with is not included in the solutions

Let’s take a look at some graphing to get the idea.

<u>Example 1</u>:

Graph.

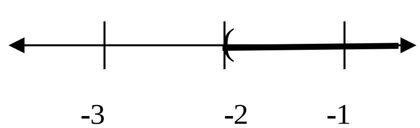
a. x > -2 b. $x \leq 3$

Solution:

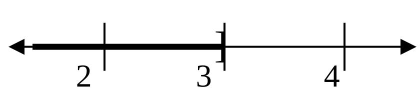
a. To graph, we need to first recognize which inequality symbol we are dealing with. In this case we are working with the “greater than” symbol. This means, the solution is all of the values greater (or larger) than -2.

   Also, since we have a regular greater than symbol, we use an open circle. The reason for this is because the -2 is not actually included in the solutions.

   So we get the graph



b. This time we want to graph all of the values “less than or equal to” three. This means we want all of the values less (or smaller) than 3 as well as the 3 itself. So we use the closed circle on the three and get the graph as follows

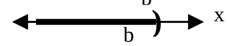


In order to simplify matters, we want to define a simple notation for inequalities. This new notation is called using intervals. There are two types of intervals on the real number line; bounded and unbounded.

Definitions:
Bounded interval- An interval with finite length, i.e. if we subtract the endpoints of the interval we get a real number.
Unbounded interval- Any interval which is not of finite length is unbounded.

Let us proceed to define these intervals by relating them to inequalities.
We start with the unbounded intervals.

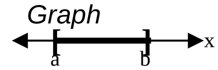
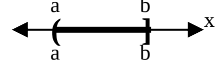
Unbounded Intervals

Inequality	*Interval Type*	*Notation*	*Graph*
$a \le x$	Half-open	$[a, \infty)$	
$a < x$	Open	(a, ∞)	
$x \le b$	Half-open	$(-\infty, b]$	
$x < b$	Open	$(-\infty, b)$	
	Entire Line	$(-\infty, \infty)$	

Notice that when writing in interval notation, we always write our intervals in increasing order. That is, we always have the smaller numbers on the left.

There are other types of inequalities that we will look at in more depth later in the section, but for the sake of organization, we will include the intervals here. These relate to the compound inequalities that we treat at the end of the section.

Bounded Intervals

Inequality	*Interval Type*	*Notation*	*Graph*
$a \le x \le b$	Closed	$[a, b]$	
$a < x < b$	Open	(a, b)	
$a \le x < b$	Half-open	$[a, b)$	
$a < x \le b$	Half-open	$(a, b]$	

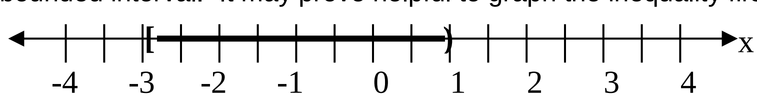
Note that the lengths of all the intervals above are $b - a$. Which is a real number and thus all the above intervals are bounded by definition.

Example 2:

Write the following in interval notation

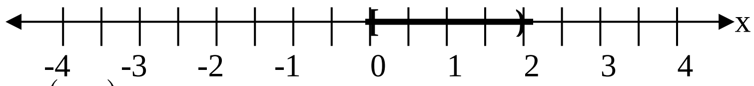
a. $-3 \le x < 1$ b. $0 < x < 2$ c. $x > -3$ d. $x \le 2$

Solution:
a. This is a bounded interval. It may prove helpful to graph the inequality first.



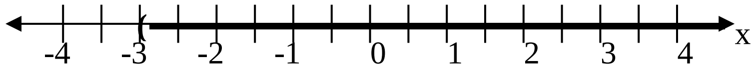
So, as an interval we get $[-3\ 1)$.

b. Again this is a bounded interval. The graph is



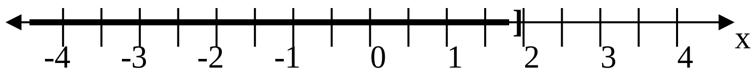
So, we get $(0,\ 2)$.

c. This is an unbounded interval. Graphing we get



Hence our interval is $(-3,\ \infty)$.

d. Finally, we have



Thus, our interval is $(-\infty,\ 2]$.

So, now that we can graph and deal with intervals, how do we solve an inequality? We need the following properties.

Properties of Inequalities
1. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
2. If $a < b$, and c is positive, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
3. If $a < b$, and c is negative, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
The rules are similar for <, $\le$ and $\ge$.

The idea is that we can add or subtract any value on both sides of an inequality symbol and nothing changes and we can multiply or divide any <u>positive</u> value on both sides of an inequality symbol and nothing changes. However, if we multiply or divide any <u>negative</u> value on both sides of an inequality symbol, we must "flip" the inequality symbol.

We use these properties to solve inequalities. Basically, all we have to remember is that we solve them just as we solved equations with the added restriction that any time we multiply or divide by a negative, we have to change the inequality symbol.

<u>Example 3</u>:

Solve and graph.

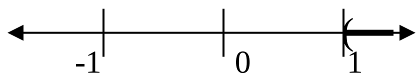
a. $-2x + 3 < 1$ b. $7x + 4 \le 2x - 6$ c. $3 - 4(x + 2) \le 6 + 4(2x + 1)$

Solution:

a. So we can simply solve this inequality as we solved equations. We just get the x alone on one side like we did with equations. The only thing we have to remember is that when we have to divide by a negative, we will need to “flip” the inequality symbol. We proceed as follows

$$\begin{array}{ll} -2x+3<1 & \\ \quad -3 \quad\; -3 & \text{Subtract 3 from both sides} \\ -2x<-2 & \\ \dfrac{-2x}{-2}>\dfrac{-2}{-2} & \text{Divide by -2, flip the inequality symbol} \\ x>1 & \end{array}$$

Now we simply graph. We (since we have a > symbol.

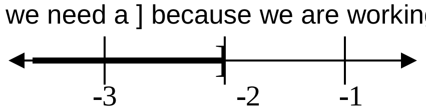


So in interval notation our solution is $(1,\infty)$.

b. Again, we solve as we did with equations and “flip” the inequality symbol if needed. We get

$$\begin{array}{ll} 7x+4\le 2x-6 & \\ -2x \qquad -2x & \text{Subtract 2x on both sides} \\ 5x+4\le -6 & \\ \quad -4 \quad -4 & \text{Subtract 4 on both sides} \\ 5x\le -10 & \\ \dfrac{5x}{5}\le\dfrac{-10}{5} & \\ x\le -2 & \end{array}$$

So we graph. Here we need a] because we are working with a ≤ symbol.



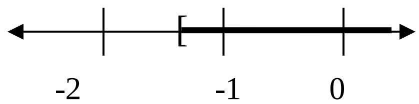
So our solution is $(-\infty,-2]$.

c. Again, we proceed as we did above. Just follow the steps that we did when we learned how to solve equations.

$$\begin{array}{ll} 3-4(x+2)\le 6+4(2x+1) & \text{Use the distributive property} \\ 3-4x-8\le 6+8x+4 & \text{Combine like terms} \\ -4x-5\le 8x+10 & \\ -8x \qquad -8x & \text{Subtract 8x on both sides} \\ -12x-5\le 10 & \\ \quad +5 \quad +5 & \text{Add 5 on both sides} \\ -12x\le 15 & \\ \dfrac{-12x}{-12}\ge\dfrac{15}{-12} & \text{Divide by -12 on both sides} \\ & \text{Reduce} \end{array}$$

$$x \geq -\frac{5}{4}$$

Now we graph. When doing so with a fractional value, we have to do our best to plot the point roughly where it should go. We have



So the solution is $\left[-\frac{5}{4}, \infty\right)$.

<u>Compound Inequalities</u>

There is another type of inequality that we need to be able to solve called in <u>compound inequality</u>. Compound inequalities come in several forms. We will only concentrate on three primary types: double inequalities, inequalities containing "and", inequalities containing "or". As we will see, the double inequalities will relate directly to our bounded intervals that we discussed at the beginning of the section.

Also, what we need to keep in mind is that when working with the "and" and "or" problems,
and means ∩ , **or** means ∪

We will illustrate how to solve all of these types in the following examples.

<u>Example 4</u>:

Solve and graph.

a. $-5 \leq 3x + 4 < 16$ b. $-2 < 5 - 4x < 1$

Solution:

a. In this example we have the so called double inequalities. In order to solve these, we want to get the x alone in the middle of the inequality symbols. We do this the same way we did in the previous example. The only difference is whatever we do one part of the inequality, we need to do to all three parts of the inequality. So we proceed as follows

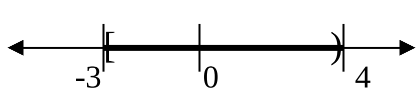
$$\begin{array}{c} -5 \leq 3x + 4 < 16 \\ -4 \qquad\quad -4 \quad\; -4 \end{array} \quad \text{Subtract 4 everywhere}$$

$$-9 \leq 3x < 12$$

$$\frac{-9}{3} \leq \frac{3x}{3} < \frac{12}{3} \quad \text{Divide by 3 everywhere}$$

$$-3 \leq x < 4$$

For the graph and solution to this inequality, all we need know is as long as the endpoints are set up in a consistent way (as they are for this one, since –3 is less than 4), then the solution is everything between the endpoints. So we have a graph of



So the solution is [-3, 4).

b. Again, we proceed as we did above. Remember, whenever multiply or divide by a negative, we must “flip” the inequality symbol.
We get

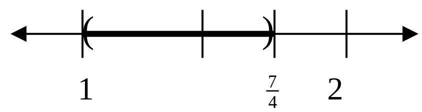
$$\begin{array}{l} -2 < 5 - 4x < 1 \\ -5 \quad -5 \qquad\ -5 \end{array}$$ Subtract 5 everywhere

$$-7 < -4x < -4$$

$$\frac{-7}{-4} > \frac{-4x}{-4} > \frac{-4}{-4}$$ Divide by -4 everywhere
(Don’t forget to “flip” the inequality symbols)

$$\frac{7}{4} > x > 1$$

Now again, the endpoints are consistent because $\frac{7}{4}$ is larger than 1 so the graph is everything between them. This gives



This gives us an interval of $\left(1, \frac{7}{4}\right)$.

Notice that whenever dealing with a double inequality, as long as the endpoints are in the correct order, the inequality will always be all values in between the endpoints.

Now we need to deal with the compound inequalities that have an “and” or an “or”. We simply need to remember that when dealing with an “and” we want only the overlapping portion of the graph, when dealing with an “or” we get to keep everything we graph.

Example 5:

Solve and graph.

a. $3x + 7 < 10 \ or \ 2x - 1 > 5$
b. $2x - 3 \geq 5 \ and \ 3x - 1 > 11$
c. $9x - 2 < 7 \ and \ 3x - 5 > 10$
d. $3x - 11 \leq 4 \ or \ 4x + 9 \geq 1$

Solution:

a. First, we can start by solving each of the inequalities separately and deal with the “or” later. We get

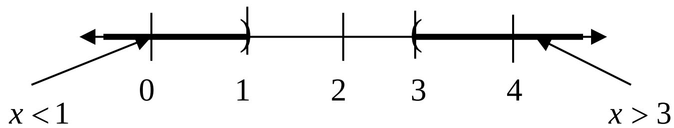
$$\begin{array}{l} 3x + 7 < 10 \quad \text{or} \quad 2x - 1 > 5 \\ \quad\ -7 \quad -7 \qquad\qquad +1 \quad +1 \end{array}$$

$$3x < 3 \ \text{ or } \ 2x > 6$$

$$\frac{3x}{3} < \frac{3}{3} \ \text{ or } \ \frac{2x}{2} > \frac{6}{2}$$

$$x < 1 \ \text{ or } \ x > 3$$

Now, to graph, we graph each of the inequalities and as stated above, since we have an "or" we get to keep everything we graph. This gives



This gives us a solution of $(-\infty, 1) \cup (3, \infty)$.

b. Again, we start by solving each inequality separately and take care of the "and" later.

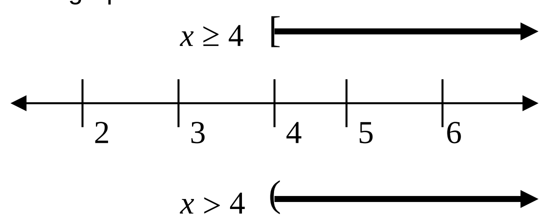
$$\begin{array}{ccc} 2x - 3 \geq 5 & \text{and} & 3x - 1 > 11 \\ \;\;+3 \;\; +3 & & \;\;+1 \;\; +1 \end{array}$$

$$2x \geq 8 \;\; \text{and} \;\; 3x > 12$$

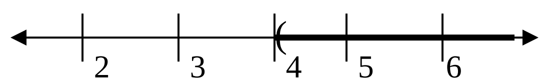
$$\frac{2x}{2} \geq \frac{8}{2} \;\; \text{and} \;\; \frac{3x}{3} > \frac{12}{3}$$

$$x \geq 4 \;\; \text{and} \;\; x > 4$$

Since this time we have an "and" we need to just graph the overlapping section. So we start by graphing each inequality individually, above or below the graph, then put the overlap onto the finished graph.



So we can see clearly that the graphs overlap 4 onward. However, at the value of 4, they do not overlap since the bottom piece has a parenthesis, and thus does not contain the value of 4. So our graph is



This means our solution is actually just the $(4, \infty)$. We need to be aware that when we are working with an "and" problem, the graph in the end will usually give us the actual solution.

Keep this in mind as we continue to work on the compound inequalities.

c. We will again proceed as before.

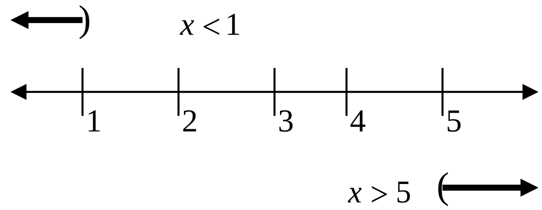
$$\begin{array}{ccc} 9x - 2 < 7 & \text{and} & 3x - 5 > 10 \\ \;\;+2 \;\; +2 & & \;\;+5 \;\; +5 \end{array}$$

$$9x < 9 \;\; \text{and} \;\; 3x > 15$$

$$\frac{9x}{9} < \frac{9}{9} \;\; \text{and} \;\; \frac{3x}{3} > \frac{15}{3}$$

$$x < 1 \;\; \text{and} \;\; x > 5$$

Since we have an “and” we want only the overlapping section. Graphing individually we get



Since there is no overlapping section, there must be no solution.

d. Lastly, we will solve as we did above.

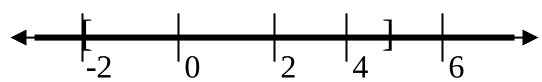
$$3x - 11 \le 4 \quad \text{or} \quad 4x + 9 \ge 1$$
$$+11 \quad +11 \qquad\qquad -9 \quad -9$$

$$3x \le 15 \quad \text{or} \quad 4x \ge -8$$

$$\frac{3x}{3} \le \frac{15}{3} \quad \text{or} \quad \frac{4x}{4} \ge \frac{-8}{4}$$

$$x \le 5 \quad \text{or} \quad x \ge -2$$

Since we have an “or” we graph both of the inequalities on the same line and we get to keep everything that we graph. We get



So since the entire number line has been shaded, all real numbers must be the solution. We write this in interval notation as $(-\infty, \infty)$.

2.4 Exercises

Graph.

1. $x > 2$
2. $x > -1$
3. $x \le -2$
4. $x \le 3$
5. $x \ge 0$
6. $x \ge 4$
7. $x < -1$
8. $x < 4$
9. $x > \frac{1}{2}$
10. $x < \frac{1}{4}$
11. $x \le -\frac{5}{3}$
12. $x \ge -\frac{3}{8}$

Solve and graph.

13. $-3x > 6$
14. $\dfrac{x}{-2} \le -3$
15. $-2 - x < 7$
16. $4x + 3 < -1$
17. $7x + 4 \ge 3x + 2$
18. $5x - 3 \le -3x + 2$
19. $5(2x + 1) - 5 > 2x$
20. $20 - 2(x + 9) \le 2(x - 5)$
21. $10 - 13(2 - x) < 5(3x - 2)$

22. $6-(2x-1)\geq 5(3x-4)$

23. $6(3-x)\geq 5-(4x-7)$

24. $3-(3-x)\geq 5-2(x-7)$

25. $\frac{3}{8}x+\frac{1}{2}<\frac{1}{4}x+2$

26. $\frac{5}{6}x-\frac{2}{3}<\frac{3}{4}x+2$

27. $\frac{1}{2}x+\frac{2}{5}>\frac{1}{4}x-\frac{2}{3}$

28. $\frac{3}{2}x-\frac{2}{7}>\frac{5}{14}x-\frac{2}{3}$

29. $3(2x-1)-(4x+1)-2(5x-6)\leq 8$

30. $3(2x-1)-(5x-4)-(7x+1)\leq -3$

31. $-5\leq 2x+1<7$

32. $-4<2x-3\leq 1$

33. $0<4x+4<7$

34. $-2<3x+7<1$

35. $3<7x-14<31$

36. $-6\leq 5x+14\leq 24$

37. $-5\leq 3x+4<16$

38. $5\leq 4x-3\leq 21$

39. $0\leq 2x-6\leq 4$

40. $5<4x-3<21$

41. $0<2x-5<9$

42. $2<3-x<3$

43. $-4<2-3(x+2)<11$

44. $-1<3-(2x-3)<0$

45. $-3<2-\frac{x}{2}\leq 1$

46. $-3<x-\frac{3}{2}\leq 3$

47. $12\geq \frac{3-x}{2}>1$

48. $1>\frac{x-4}{-3}>-2$

49. $3x+7>-2 \textit{ or } 3x+7<-4$

50. $6x+5<11 \textit{ or } 3x-1>8$

51. $x+3\geq 6 \textit{ and } 2x\geq 8$

52. $3x+1<7 \textit{ and } 3x+5\geq -1$

53. $x+4\geq 5 \textit{ and } 2x\geq 6$

54. $9x-2<7 \textit{ or } 3x-5>10$

55. $4x-1>11 \textit{ or } 4x-1\leq -11$

56. $2x-3\leq 5 \textit{ and } 3x-1>11$

57. $6x+5<-1 \textit{ or } 1-2x<7$

58. $9-x\geq 7 \textit{ and } 9-2x<3$

59. $3x+1<7 \textit{ or } 3x+5\geq -1$

60. $5x-3<-18 \textit{ or } 6x-1\geq 17$

61. $3x-1\leq 8 \textit{ and } 6x+5<11$

62. $3x-1<-19 \textit{ or } 2x+4\geq 16$

63. $5x-3\leq -18 \textit{ or } 1-6x<-17$

64. $3x-11<4 \textit{ or } 4x+9\geq 1$

65. $3-7x\leq 31 \textit{ and } 5-4x>1$

66. $8x+2\leq -14 \textit{ and } 4x-2>10$

67. $2x-3>5 \textit{ and } 3x-1>11$

68. $3x-5>10 \textit{ or } 3x-5<-10$

69. $1-4x<-11 \textit{ or } 1-4x>11$

70. $1-3x<16 \textit{ and } 1-4x\geq 5$