

15.4 The Binomial Theorem

In this section we want to spend time discussing the expansion of binomials. This turns out to be very useful in many different higher math classes, including statistics and calculus. Many of the details of how this is used are reserved for those classes. Our purpose here is to simply give a introduction to the binomial theorem.

The Binomial Theorem

The expansion of $(x + y)^n$ is given by

$$(x + y)^n = x^n + nx^{n-1}y + \binom{n}{r}x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Even though the formula seems pretty involved, it is not as bad as it seems. There are definite patterns to follow for both the coefficients as well as powers.

Clearly, the reason that we talk about the binomial theorem and binomial expansion within a chapter discussing sequences and series is that the binomial theorem is a finite series.

The first thing we need to get used to is the expression $\binom{n}{r}$.

Example 1:

Evaluate.

a. $\binom{6}{2}$

b. $\binom{9}{6}$

c. $\binom{5}{0}$

d. $\binom{10}{10}$

Solution:

a. To evaluate, its just a matter of using the formula.

$$\begin{aligned}\binom{6}{2} &= \frac{6!}{(6-2)!2!} \\ &= \frac{6!}{4!2!} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2} \\ &= \frac{30}{2} \\ &= 15\end{aligned}$$

b. Again, using the formula we have

$$\begin{aligned}\binom{9}{6} &= \frac{9!}{(9-6)!6!} \\ &= \frac{9!}{3!6!} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\end{aligned}$$

$$= \frac{504}{6}$$

$$= 84$$

c. By the formula we have

$$\binom{5}{0} = \frac{5!}{(5-0)!0!}$$

$$= \frac{5!}{5!}$$

$$= 1$$

d. Lastly,

$$\binom{10}{10} = \frac{10!}{(10-10)!10!}$$

$$= \frac{10!}{0!10!}$$

$$= 1$$

As it turns out, there is a much simpler way to determine the values of the coefficients of a binomial expansion. It is called **Pascal's Triangle**, named for the French mathematician Blaise Pascal. It is found by starting with a row of the single digit 1 then a row of the digits 1 and 1. Then, each subsequent row is generated by starting and ending with a 1 and adding the two consecutive values from the previous row. We get

$$\begin{array}{c} 1 \\ 1 \ 1 \\ 1 \ 2 \ 1 \\ 1 \ 3 \ 3 \ 1 \\ 1 \ 4 \ 6 \ 4 \ 1 \\ 1 \ 5 \ 10 \ 10 \ 5 \ 1 \\ 1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1 \\ 1 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 1 \\ \text{Etc.} \end{array}$$

As it turns out, these numbers correspond to the coefficients of the binomial theorem starting with the top row being used for $n = 0$ in the binomial theorem.

Lets see exactly how this works.

Example 2:

Use Pascal's Triangle to expand.

a. $(x + y)^4$

b. $(a + b)^6$

Solution:

- a. There are two parts to expanding a binomial: the coefficients and the variables. To determine the coefficients, we just need to use Pascal's Triangle. As stated above, the top row is for $n = 0$ and so since $n = 4$ in this problem, we need the 5th row down. So our coefficients will be 1 4 6 4 1.

Next, to produce the variable parts, we just need to start with x^4y^0 and decrease our power of x while increasing the power of y , one at a time until we end with x^0y^4 .

So our pattern would be $x^4y^0 + x^3y^1 + x^2y^2 + x^1y^3 + x^0y^4$.
So, putting the proper coefficients in and simplifying we get

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- b. This time we choose the 7th row on Pascal's Triangle and use the pattern for variables that we used in part a. Starting with a^6b^0 .
This gives us

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Although this seems very simple, we have to keep in mind that this will clearly only work in the situations in which the coefficients are 1.

So, how do we use the binomial theorem as well as Pascal's Triangle to expand binomials when the coefficients are not 1. It's just a matter of making a substitution as we will see in the next example.

Example 3:

Expand the binomial.

a. $(x + 1)^3$

b. $(y - 2)^4$

c. $\left(\frac{2}{x} + 3y\right)^5$

Solution:

- a. The best way to deal with expanding when the coefficients and variables are not set up exactly as the binomial theorem states is to apply the theorem and Pascal's Triangle to the terms as they are and simplify the powers and products that result.

So in this case, we would have

$$\begin{aligned} & x^31^0 + 3x^21^1 + 3x^11^2 + x^01^3 \\ & = x^3 + 3x^2 + 3x + 1 \end{aligned}$$

- b. As we did in part a, we will just apply the theorem and Pascal's Triangle to the terms as they are. We just need to be very careful of the signs.

We get

$$\begin{aligned} & y^4(-2)^0 + 4y^3(-2)^1 + 6y^2(-2)^2 + 4y^1(-2)^3 + y^0(-2)^4 \\ & = y^4 + 4y^3(-2) + 6y^2 \cdot 4 + 4y \cdot -8 + 16 \\ & = y^4 - 8y^3 + 24y^2 - 32y + 16 \end{aligned}$$

- c. Lastly, this time it's much more difficult. But we will still use the same principles that we have used in the first two parts. We will just insert $\frac{2}{x}$ and $3y$ into the theorem and carefully simplify.

$$\begin{aligned} & \left(\frac{2}{x}\right)^5 (3y)^0 + 5\left(\frac{2}{x}\right)^4 (3y)^1 + 10\left(\frac{2}{x}\right)^3 (3y)^2 + 10\left(\frac{2}{x}\right)^2 (3y)^3 + 5\left(\frac{2}{x}\right)^1 (3y)^4 + \left(\frac{2}{x}\right)^0 (3y)^5 \\ & = \frac{32}{x^5} + 5 \cdot \frac{16}{x^4} \cdot 3y + 10 \cdot \frac{8}{x^3} \cdot 9y^2 + 10 \cdot \frac{4}{x^2} \cdot 27y^3 + 5 \cdot \frac{2}{x} \cdot 81y^4 + 243y^5 \\ & = \frac{32}{x^5} + \frac{240y}{x^4} + \frac{720y^2}{x^3} + \frac{1080y^3}{x^2} + \frac{810y^4}{x} + 243y^5 \end{aligned}$$

15.4 Exercises

Evaluate.

- | | | | | |
|---------------------|--------------------|---------------------|--------------------|---------------------|
| 1. $\binom{4}{2}$ | 2. $\binom{6}{3}$ | 3. $\binom{10}{4}$ | 4. $\binom{12}{3}$ | 5. $\binom{28}{27}$ |
| 6. $\binom{30}{20}$ | 7. $\binom{64}{4}$ | 8. $\binom{15}{15}$ | 9. $\binom{12}{0}$ | 10. $\binom{10}{8}$ |

Expand the binomial.

- | | | | |
|---------------------------------------|--|--|--|
| 11. $(x + y)^3$ | 12. $(a + b)^2$ | 13. $(x + y)^4$ | 14. $(a + b)^3$ |
| 15. $(a + b)^5$ | 16. $(x + y)^5$ | 17. $(a + b)^6$ | 18. $(x + y)^8$ |
| 19. $(x + y)^7$ | 20. $(a + b)^8$ | 21. $(x - 3)^3$ | 22. $(x - 1)^3$ |
| 23. $(x + 4)^4$ | 24. $(x + 5)^4$ | 25. $(2 - a)^5$ | 26. $(1 - x)^6$ |
| 27. $(3x - 1)^4$ | 28. $(4a - 1)^4$ | 29. $(2a + 1)^3$ | 30. $(3x + 2)^3$ |
| 31. $(2x + 3y)^5$ | 32. $(3a + 2b)^4$ | 33. $(x - 4y)^5$ | 34. $(a - 3b)^5$ |
| 35. $(a + 4b)^6$ | 36. $(3x - 2a)^5$ | 37. $\left(\frac{1}{x} + \frac{1}{y}\right)^5$ | 38. $\left(\frac{2}{a} + \frac{3}{b}\right)^4$ |
| 39. $\left(7a - \frac{1}{x}\right)^7$ | 40. $\left(\frac{2}{x} - \frac{3}{y}\right)^7$ | | |