15.3 Geometric Sequences and Series

The other type of sequence and series we wanted to take a closer look at is the geometric sequence and series. Let's start with a definition of the geometric sequence.

Definition: **Geometric Sequence**- a sequence is geometric if the ratio of consecutive terms is constant. Therefore,

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \cdots \qquad r \neq 0$$

The number *r* is called the common ratio.

We can see this is very similar to the arithmetic sequence except this time, the consecutive terms are off by a constant multiple, the common ratio.

Example 1:

Find the common ratio.

a. 5, 15, 45, 135, …

b. $a_n = 3(2^n)$

c. $a_n = \left(-\frac{2}{3}\right)^n$

Solution:

a. To find the common ratio, all we need to do is divide consecutive terms of the sequence and make sure we always get the same number.
So we get

$$\frac{15}{5} = 3$$
$$\frac{45}{15} = 3$$
$$\frac{135}{45} = 3$$

Therefore, the common ratio is 3.

b. Here, since we don't have the specific terms, but instead, the general form of the terms it's a little more difficult, but we proceed the same way.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{3(2^{n+1})}{3(2^n)} \\ &= \frac{3(2^n \cdot 2^1)}{3(2^n)} \\ &= 2\end{aligned}$$

So, our common ratio is 2.

c. Again, we will proceed as we did in part b.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\left(-\frac{2}{3}\right)^{n+1}}{\left(-\frac{2}{3}\right)^n} \\ &= \frac{\left(-\frac{2}{3}\right)^n \cdot \left(-\frac{2}{3}\right)^1}{\left(-\frac{2}{3}\right)^n}\end{aligned}$$

$$= -\frac{2}{3}$$

Therefore, the common ratio is $-\frac{2}{3}$.

Again, as we did with arithmetic sequences we also want to be able to find the n-th term. So, we have the following

n-th Term of a Geometric Sequence
The n-th term of a geometric sequence is $$a_n = a_1 r^{n-1}$$ where *r* is the common ratio.

Because of this, every geometric sequence can be written as

$$a_1,\ \ a_1 r,\ \ a_1 r^2,\ \ \ a_1 r^3,\ \ a_1 r^4,\ \ \cdots$$

Example 2:

Find the n-th term.

a. $a_1 = 3, r = 2$

b. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

c. $a_4 = 125, a_{10} = \frac{125}{64}$

Solution:

a. Finding the n-th term in this case is as simple as putting the values into the formula above. This gives

$$a_n = 3 \cdot 2^{n-1}$$

b. This time, it is a little more complex simply because we need to find the common ratio, *r*, first. However, as we saw in example 1, we just need to divide consecutive terms. This gives

$$\frac{\frac{1}{2}}{1} = \frac{1}{2},\ \ \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{1}{2},\ \ etc$$

So, $r = \frac{1}{2}$. Therefore, $a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-1}$.

c. Finally, here we need to find both the common ratio, *r*, as well as our first term, a_1. The first thing we notice is that the given terms are 6 terms apart in the sequence. Since each consecutive term is multiplied by the common ratio, we know the following

$$a_{10} = a_4 r^6$$
$$\frac{125}{64} = 125 r^6$$
$$\frac{1}{64} = r^6$$
$$r = \frac{1}{2}$$

Now we can find a_1 by using the n-th term formula as follows

$$a_n = a_1 r^{n-1}$$
$$a_4 = a_1 \left(\frac{1}{2}\right)^{4-1}$$

$$125 = a_1\left(\frac{1}{2}\right)^3$$
$$125 = \frac{1}{8}a_1$$
$$a_1 = 1000$$

So, our n-th term is

$$a_n = 1000\left(\frac{1}{2}\right)^{n-1}$$

Clearly, the sum of the terms of a geometric sequence would become a geometric series. And, therefore, we want to know the sum of this series.

The Sum of a Finite Geometric Series

The sum of the geometric series $\sum_{i=1}^{n} a_1 r^{i-1}$ is

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

where *r* is the common ratio.

The proof of this formula omitted for the sake of brevity. A proof can be found in most calculus textbooks.

More importantly, though, is that in order for this formula to work, **the index of summation must start with *i* = 1** and **the power on the common ration must be *i* – 1**. If it is anything else, we must make an adjustment to either, or both, values before using the formula.

Example 3:

Find the sum.

a. $\sum_{n=1}^{7} 64\left(-\frac{1}{2}\right)^{n-1}$ b. $\sum_{n=1}^{10} 3(2)^n$ c. $\sum_{i=0}^{6} 5(1.02)^i$

Solution:

a. Since this sum is already in the proper form, we simply need to plug in the values of $a_1 = 64$, $n = 7$ and $r = -\frac{1}{2}$ into the formula. We get

$$S_7 = 64\left(\frac{1-\left(-\frac{1}{2}\right)^7}{1-\left(-\frac{1}{2}\right)}\right)$$
$$= 64\left(\frac{1-\left(-\frac{1}{128}\right)}{1-\left(-\frac{1}{2}\right)}\right)$$
$$= 64\left(\frac{\frac{129}{128}}{\frac{3}{2}}\right)$$
$$= 64\left(\frac{43}{64}\right)$$
$$= 64$$

So, the sum is 64.

b. This time we need to notice that the power is not correct in our sigma notation. In order to use the formula, the power must be n – 1. So, to take care of this, we need to adjust the power by using the properties of exponents. Namely, we need to recall that when we multiply two expressions of the same base, we need to add the exponents.
In other words, $2^{n-1}2^1 = 2^{n-1+1} = 2^n$.
So we write

$$\sum_{n=1}^{10} 3(2)^n = \sum_{n=1}^{10} 3(2)^{n-1} \cdot 2^1$$
$$= \sum_{n=1}^{10} 6(2)^{n-1}$$

Now we can proceed to find the sum using the formula.

$$S_{10} = 6\left(\frac{1-2^{10}}{1-2}\right)$$
$$= 6\left(\frac{1-1024}{-1}\right)$$
$$= 6 \cdot \frac{-1023}{-1}$$
$$= 6138$$

c. As in part b, we need to adjust the power on the common ratio, but we also need to notice that the index of summation doesn't start with 1 as it should. This means we will have to adjust that as well. In this case, though, that just means we need to write out the *i* = 0 term first, followed by the rest of the series.
So we have

$$\sum_{i=0}^{6} 5(1.02)^i = 5(1.02)^0 + \sum_{i=1}^{6} 5(1.02)^i$$
$$= 5 + \sum_{i=1}^{6} 5 \cdot 1.02^1 \cdot (1.02)^{i-1}$$
$$= 5 + \sum_{i=1}^{6} 5.1 \cdot (1.02)^{i-1}$$

Now we just need to find $\sum_{i=1}^{6} 5.1 \cdot (1.02)^{i-1}$. We get

$$S_6 = 5.1\left(\frac{1-1.02^6}{1-1.02}\right)$$
$$= 5.1\left(\frac{1-1.02^6}{1-1.02}\right)$$
$$\approx 5.1(6.308)$$
$$\approx 32.171$$

Putting this in to get the finished sum we have

$$\sum_{i=0}^{6} 5(1.02)^i = 5 + \sum_{i=1}^{6} 5.1 \cdot (1.02)^{i-1}$$
$$\approx 5 + 32.171$$
$$\approx 37.171$$

Lastly, on a geometric series, we also want to be able to find the sum of an infinite series. This proves to be quite helpful in higher math classes. So, we have the following formula, given without proof.

The Sum of a Infinite Geometric Series
If $\lvert r \rvert < 1$, then the sum of the geometric series $\sum_{i=1}^{\infty} a_1 r^{i-1}$ is $S = \frac{a_1}{1-r}$ where *r* is the common ratio.

Notice, this only works when the value of r is less than 1. Also, as in the finite sum, make sure that the sum starts with 1 and the power is i-1.

Example 4:

Find the sum.

a. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

b. $\sum_{n=1}^{\infty} 4\left(\frac{1}{4}\right)^{n-1}$

c. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}, \cdots$

Solution:

a. To find the infinite sum, we just need to put the values of $a_1 = 1$ and $r = \frac{1}{2}$ into the formula.

$$S = \frac{a_1}{1-r}$$
$$= \frac{1}{1-\frac{1}{2}}$$
$$= \frac{1}{\frac{1}{2}}$$
$$= 2$$

b. Again, this problem is fairly straight forward. Plugging in we get

$$S = \frac{a_1}{1-r}$$
$$= \frac{4}{1-\frac{1}{4}}$$
$$= \frac{4}{\frac{3}{4}}$$

$$= \frac{16}{3}$$

c. Finally, we need to start by getting the sum into the proper form. Meaning, we want the sum in sigma notation so that we can easily find what we need for our formula.
By dividing consecutive terms we can see that our common ratio is $\frac{2}{3}$.

Since the sum starts with our common ratio, we must have the following sum

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

As before, we need to adjust the power so that we have a power of n-1. This gives us

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{2}{3} \cdot \left(\frac{2}{3}\right)^{n-1}$$

Now we just plug our values into the sum formula

$$S = \frac{\frac{2}{3}}{1 - \frac{2}{3}}$$

$$= \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$= 2$$

Even though, in the last example, we clearly knew the first term in the sum, we went through the motions of putting the sum into its proper sigma notation because it is always a good practice for verification purposes.

15.3 Exercises

Find the common ratio.

1. 3, 9, 27, 81, …

2. 12, 36, 108, 324, …

3. 5, -25, 125, -625, …

4. 1, -2, 4, -8, …

5. $a_n = 2(3^n)$

6. $a_n = 5(7^n)$

7. $a_n = (-1.1)^{n+1}$

8. $a_n = 4(-4)^n$

9. $a_n = 6\left(-\frac{3}{4}\right)^n$

10. $a_n = \left(\frac{3}{2}\right)^n$

Find the n-th term.

11. $a_1 = 4,\ r = 2$

12. $a_1 = 5,\ r = 6$

13. $a_1 = 3,\ r = 3$

14. $a_1 = 2,\ r = -2$

15. $a_1 = 1,\ r = -\frac{1}{2}$

16. $a_1 = -3,\ r = -1$

17. 2, 10, 50, 250, …

18. 5, 15, 45, …

19. 1, -2, 4, -8, …

20. -2, 6, -18, -54, …

21. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

22. $1, -\frac{1}{5}, \frac{1}{25}, -\frac{1}{125}, \ldots$

23. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

24. $5, \frac{5}{4}, \frac{5}{16}, \frac{5}{64}, \ldots$

25. $a_3 = 54,\ a_5 = 486$

26. $a_2 = 3,\ a_6 = 243$

27. $a_3 = 64,\ a_6 = -4096$

28. $a_2 = -18,\ a_5 = \frac{2}{3}$

29. $a_4 = \frac{128}{27},\ a_8 = \frac{32768}{2187}$

30. $a_3 = \frac{16}{3},\ a_5 = \frac{64}{27}$

Find the sum.

31. $\sum_{n=1}^{9} 2^{n-1}$

32. $\sum_{n=1}^{8} 3^{n-1}$

33. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$

34. $\sum_{i=1}^{9} 8\left(-\frac{1}{3}\right)^{i-1}$

35. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$

36. $\sum_{n=0}^{15} 2\left(\frac{2}{3}\right)^{n}$

37. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$

38. $\sum_{i=1}^{11} \left(-\frac{5}{2}\right)^{i}$

39. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$

40. $\sum_{i=0}^{16} 5\left(\frac{2}{3}\right)^{i-1}$

41. $\sum_{n=0}^{5} 300(1.06)^{n}$

42. $\sum_{n=0}^{4} 150(1.11)^{n}$

43. $\sum_{i=1}^{\infty} 4\left(\frac{3}{5}\right)^{i-1}$

44. $\sum_{i=1}^{\infty} 2\left(\frac{1}{2}\right)^{i-1}$

45. $\sum_{n=1}^{\infty} 3\left(\frac{1}{10}\right)^{n-1}$

46. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n}$

47. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

48. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$

49. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

50. $-1 + \frac{2}{5} - \frac{4}{25} + \frac{8}{125} + \cdots$