15.2 Arithmetic Sequences and Series

Now that we have a basic idea of the concepts of sequences and series, in the next two sections, we will look at two very specific types of sequences and series. We start with the Arithmetic type.

> **<u>Definition</u>**: **<u>Arithmetic Sequence</u>**- a sequence is arithmetic if it has a constant difference between consecutive terms.
>
> Therefore, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is arithmetic if there is a number, *d*, such that
>
> $$a_2 - a_1 = d, \;\; a_3 - a_2 = d, \;\; a_4 - a_3 = d, \cdots$$
>
> The number *d* is called the <u>common difference</u>.

So, a sequence is arithmetic if when you subtract consecutive terms, you always get the same number. That particular number is the common difference.

<u>Example 1</u>:

Find the common difference in the sequence.

a. 6, 9, 12, 15, … b. $a_n = 2 - 5n$ c. $a_n = \frac{1}{2}(n+2)$

Solution:

a. Finding the common difference is as simple as subtracting consecutive terms and seeing if we get the same value. So we have

9 – 6 = 3
12 – 9 = 3
15 – 12 = 3

Therefore, the common difference is 3

b. There are a couple of different ways to do this particular problem. Since our sequence is given in general, we should probably do the problem in general.
So, consecutive terms would be, $a_n = 2 - 5n$ and $a_{n+1} = 2 - 5(n+1)$.
Subtracting we get

$$\begin{aligned} a_{n+1} - a_n &= 2 - 5(n+1) - (2 - 5n) \\ &= 2 - 5n - 5 - 2 + 5n \\ &= -5 \end{aligned}$$

So the common difference must be -5

c. Again, we will proceed like we did in part b. Consecutive terms would be

$$a_n = \frac{1}{2}(n+2) \text{ and } a_{n+1} = \frac{1}{2}((n+1)+2)$$

Subtracting we get

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{2}(n+3) - \frac{1}{2}(n+2) \\ &= \frac{1}{2}n + \frac{3}{2} - \frac{1}{2}n - 1 \\ &= \frac{1}{2} \end{aligned}$$

As we have seen in the last section, its also very important to be able to find the n-th term of a sequence. In an arithmetic sequence, we have a formula for finding the n-th term. The proof of this formula is fairly straight forward by writing out terms and is left to the reader and can also be inferred from the results of example 1.

n-th Term of an Arithmetic Sequence
The n-th term of an arithmetic sequence is $$a_n = a_1 + (n-1)d$$ where *d* is the common difference.

Example 2:

Find the n-th term.

a. $a_1 = 15, d = 4$ b. $a_1 = -y, d = 5y$ c. 10, 5, 0, -5, -10, … d. $a_4 = 16, a_{10} = 46$

Solution:

a. Plugging the values directly into the formula gives us

$$\begin{aligned} a_n &= 15 + (n-1) \cdot 4 \\ &= 15 + 4n - 4 \\ &= 11 + 4n \end{aligned}$$

So, $a_n = 11 + 4n$.

b. Again, plugging in gives

$$\begin{aligned} a_n &= -y + (n-1) \cdot 5y \\ &= -y + 5ny - 5y \\ &= -4y + 5ny \end{aligned}$$

So, $a_n = -4y + 5ny$.

c. The first thing we need to do determine the common difference. Subtracting consecutive terms clearly gives us $d = -5$. So we have

$$\begin{aligned} a_n &= 10 + (n-1) \cdot -5 \\ &= 10 - 5n + 5 \\ &= 15 - 5n \end{aligned}$$

So, $a_n = 15 - 5n$.

d. This is much more complicated. The first thing we need to do is determine our a_1 and our d values. Since we have been given a_4 and a_{10} we know that there are 6 terms of the sequence between these two values. Each term is increased by the common difference *d*. So, therefore we know

$$a_{10} = a_4 + 6d$$

Substituting the values and solving gives

$$\begin{aligned} 46 &= 16 + 6d \\ 30 &= 6d \\ d &= 5 \end{aligned}$$

Now that we have *d* we can find a_1 by using the general formula for a_n. We proceed as follows

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_4 &= a_1 + (4-1) \cdot 5 \\ 16 &= a_1 + 3 \cdot 5 \\ 16 &= a_1 + 15 \\ a_1 &= 1 \end{aligned}$$

Therefore, we can now produce our formula for a_n. We get

$$\begin{aligned} a_n &= 1 + (n-1) \cdot 5 \\ &= 1 + 5n - 5 \\ &= -4 + 5n \end{aligned}$$

So, $a_n = -4 + 5n$.

Now that we know what an arithmetic sequence is, let's turn our attention to the arithmetic series.

Clearly, an arithmetic series is just the sum of an arithmetic sequence. Therefore, it is only important for us to be able to find the sum of an arithmetic series.

The Sum of a Finite Arithmetic Series
The sum of a finite arithmetic series, $\sum_{i=1}^{n} a_i$ is given by $$S_n = \frac{n}{2}(a_1 + a_n)$$

Using this formula is fairly straight forward as we can see in the next example.

Example 3:

Find the sum.

a. $\sum_{n=1}^{50} n$

b. $\sum_{n=1}^{98} 5n$

c. $\sum_{n=1}^{300} (2n-1)$

d. 1+2+3+…+100

e. 8+20+32+44+…, n=10

f. -6 – 2 + 2 + 6 + …, n=50

Solution:

a. To find the sum, we just need the first term and the 50th term. Calculating we get

$$a_1 = 1, \qquad a_{50} = 50$$

Plugging into the formula for the sum gives us

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_{50} &= \frac{50}{2}(1 + 50) \\ &= 25(51) \\ &= 1275 \end{aligned}$$

b. As is part a, we just need to find the first term, which is clearly 5, and the 98th term, which is $5 \cdot 98 = 490$. Using the formula gives us

$$\begin{aligned} S_{98} &= \frac{98}{2}(5 + 490) \\ &= 49(495) \\ &= 24255 \end{aligned}$$

c. Again, we find $a_1 = 2(1) - 1 = 1$ and $a_{300} = 2(300) - 1 = 599$. Plugging in we get

$$\begin{aligned} S_{300} &= \frac{300}{2}(1 + 599) \\ &= 150(600) \\ &= 90000 \end{aligned}$$

d. To find the sum here, we just need to put the values into the formula. Here, $a_1 = 1$ and $a_{100} = 100$. So the sum is

$$\begin{aligned} S_{100} &= \frac{100}{2}(1 + 100) \\ &= \frac{10100}{2} \\ &= 5050 \end{aligned}$$

e. This time we do not know what the 10th term of the series is, nor do we have a formula to determine it. However, we can use the concepts we have learned to determine the sigma notation of the series and then we will have the ability to find everything we need.

To do this, we need to start with finding the common difference, which is clearly 12. Then, as we did in example 2, we find a_n by the formula as follows

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_n &= 8 + (n-1)\cdot 12 \\ a_n &= -4 + 12n \end{aligned}$$

So our series is $\sum_{n=1}^{10}(-4 + 12n)$. Also, to find the sum we will need the 10th term. Using our sigma notation we see $a_{10} = -4 + 12(10) = 116$.

Now, using our formula to find the sum gives us

$$\begin{aligned} S_{10} &= \frac{10}{2}(8 + 116) \\ &= 5(124) \\ &= 620 \end{aligned}$$

f. Again, we need to start with finding our common difference, sigma notation and our 50th term.

Subtracting to get the common difference we get $d = 4$.
So,

$$\begin{aligned} a_n &= -6 + (n-1)\cdot 4 \\ a_n &= -10 + 4n \end{aligned}$$

So the series is $\sum_{n=1}^{50}(-10 + 4n)$.

Therefore, our 50th term is $a_{50} = -10 + 4(50) = 190$.

This means our sum is

$$\begin{aligned} S_{50} &= \frac{50}{2}(-6 + 190) \\ &= 25(184) \\ &= 4600 \end{aligned}$$

15.2 Exercises

Find the common difference.

1. 2, 4, 6, …
2. 3, 6, 9, 12, …
3. 2, -3, -8, -13, …
4. 8, 6, 4, 2, …

5. $a_n = 8 + 4n$ 6. $a_n = 3 - 2n$ 7. $a_n = 7 - 5n$ 8. $a_n = -1 + 4n$

9. $a_n = \frac{1}{4}(n+3)$ 10. $a_n = \frac{1}{3}(n-2)$

Find the n-th term.

11. $a_1 = 8,\ d = 2$ 12. $a_1 = 5,\ d = 6$ 13. $a_1 = 0,\ d = -3$ 14. $a_1 = 10,\ d = -4$

15. $a_1 = x,\ d = 2x$ 16. $a_1 = 2y,\ d = -5y$

17. 4, 8, 12, 16, … 18. 2, 8, 14, 20, … 19. -2, 1, 4, 7, …

20. -6, -2, 2, 6, … 21. -3, -1, 1, 3, … 22. 40, 37, 34, 31, …

23. $a_2 = 8,\ a_5 = 14$ 24. $a_3 = 9,\ a_6 = 21$ 25. $a_3 = 94,\ a_6 = 85$

26. $a_4 = 16,\ a_{10} = 46$ 27. $a_5 = 190,\ a_{10} = 115$ 28. $a_8 = 26,\ a_{12} = 42$

29. $a_6 = -63,\ a_{15} = -153$ 30. $a_3 = 19,\ a_{15} = -1.1$

Find the sum.

31. $\sum_{i=1}^{100} 2i$ 32. $\sum_{k=1}^{50} 5k$ 33. $\sum_{n=1}^{500}(n+3)$ 34. $\sum_{n=1}^{100}(2n-3)$

35. $\sum_{i=1}^{20}(2i+5)$ 36. $\sum_{i=1}^{50}(6i-30)$ 37. $\sum_{n=1}^{250}(1000-n)$ 38. $\sum_{i=1}^{500}(2i-500)$

39. $\sum_{n=1}^{100}\frac{n+4}{2}$ 40. $\sum_{i=1}^{100}\frac{2i-3}{2}$

41. 1+2+3+ … , n = 19 42. 2+4+6+8+ … , n = 30

43. 40+37+34+31+ … , n = 10 44. 31+27+23+19+ … , n = 12

45. 5+16+37+38+ … , n = 150 46. 9+19+29+39+ … , n = 70

47. 4+4+4+4+ … , n = 1000 48. -3+0+3+6+9+ … , n = 50

49. 17+14+11+8+ … , n = 100 50. -1+1+3+5+ … , n = 100