15.1 Sequences and Series Basics

In this final chapter, we want to take an introductory look at the concepts of sequences and series. For a more detailed discussion these topics, the reader should take a second semester calculus course. Lets start with a basic definition.

> **Definition**: **Sequence**- a function whose domain is the set of positive integers. The function values $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the terms of the sequence.
>
> A sequence of infinitely many terms is an infinite sequence and finitely many terms is a finite sequence.

Example 1:

Find the first four terms of the sequence.

a. $a_n = 2n - 3$ b. $a_n = \frac{(-1)^n}{2n+1}$ c. $a_1 = 3,\ \ a_{k+1} = 2(a_k - 1)$

Solution:

a. To find the first four terms, we simply need to evaluate the expression for n=1, 2, 3 and 4. So we get

$$a_1 = 2(1) - 3 = -1$$
$$a_2 = 2(2) - 3 = 1$$
$$a_3 = 2(3) - 3 = 2$$
$$a_4 = 2(4) - 3 = 5$$

b. Again, substituting n= 1, 2, 3 and 4 we get

$$a_1 = \frac{(-1)^1}{2(1)+1} = \frac{-1}{3}$$
$$a_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$$
$$a_3 = \frac{(-1)^3}{2(3)+1} = \frac{-1}{7}$$
$$a_4 = \frac{(-1)^4}{2(4)+1} = \frac{1}{9}$$

c. Lastly, this sequence is different than the other two. This time, each term of the sequence is based on the previous term. So we start with the first term (which we are given) and generate the other terms one at a time.

$$a_1 = 3$$
$$a_2 = 2(a_1 - 1) = 2(3 - 1) = 2(2) = 4$$
$$a_3 = 2(a_2 - 1) = 2(4 - 1) = 2(3) = 6$$
$$a_4 = 2(a_3 - 1) = 2(6 - 1) = 2(5) = 10$$

The sequence in part c. above (where the terms are generated by previous terms) is referred to as a recursive sequence.

Finding the terms of a sequence given its n-th terms (as in the first example) is fairly simple. Finding the n-th term based upon the terms (in other words, going backwards) is more complex, as we will see in the next example.

Example 2:

Find the n-th term of the sequence.

a. 1, 4, 7, 10, 13, . . . b. 0, 3, 8, 15, 24, . . . c. 1, $-\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

Solution:

a. From example 1 above, we can see that there are sometime a couple of ways to express a sequence. That is, either by the n-th term of the sequence, or, occasionally, by using a recursive sequence. When deciding, it best to go with whichever type you find more simple. In this case, we can clearly see that each term of the sequence is 3 more than the previous term. So, the simplest way to express this sequence would be

$$a_1 = 1, \;\; a_{k+1} = a_k + 3$$

The other option would require us to notice that when n = 1, we get 1, then when n = 2 we get 4. So by a little trial and error we could see that $a_n = 3n - 2$. Then we can make sure that this formula works for the other seen values of n, which is clearly does.

So in this case, either answer is acceptable.

b. This time it is much more difficult to produce the formula for a_n. We can see that there is no way to make the sequence reclusively. Each term is not uniformly generated from the previous term.

However, it looks like the terms are one less than the square of n. This is, by no means, obvious. But, the more we work with sequences, the more we are able to detect these patterns. In any case we have

$$a_n = n^2 - 1$$

c. Finally, we can clearly see a pattern for the denominators. That is, clearly we have each term is $\frac{1}{n^2}$. The issue here is that the sign alternates between positive and negative.

Looking back at example 1 part b, we see that the sign on the values there alternated as well. This alternating sign is produced from either one of two possible expressions, $(-1)^n$ or $(-1)^{n+1}$.

To decide which one we have, we can simply try a few of the first values of the sequence to see which one fits. In this case, the expression $(-1)^{n+1}$ will result in a positive for n = 1, negative for n = 2, positive for n = 3, etc. This matches our values in the sequence.

Therefore, our sequence must be

$$a_n = \frac{(-1)^{n+1}}{n^2}$$

A very common function used in sequences is something called the factorial.

Definition: **Factorial**- if n is a positive integer, n factorial, written $n!$, is defined by

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n$$

Also, 0! is defined to be 1.

In other words, the factorial of a number is the product of the number, and every number smaller than the number.

Example 3:

Evaluate.

a. $\frac{4!}{7!}$ b. $\frac{2!\cdot 5!}{3!\cdot 4!}$ c. $\frac{n!}{(n+1)!}$ d. $\frac{(2n-1)!}{(2n+1)!}$

Solution:

a. The best way to simplify an expression with multiple factorials is write out the values and see what cancel, if anything. We get

$$\begin{aligned}\frac{4!}{7!} &= \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 4}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5\cdot 6\cdot 7}\\ &= \frac{1}{5\cdot 6\cdot 7}\\ &= \frac{1}{900}\end{aligned}$$

b. Again, we will write it out and look for cancelling.

$$\begin{aligned}\frac{2!\cdot 5!}{3!\cdot 4!} &= \frac{1\cdot 2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdot 4}\\ &= \frac{5}{3}\end{aligned}$$

c. This time, we will have to be more general when we write out the values since we do not have the value for n. However, keep in mind, n+1 is simply the next number after n. So we get

$$\frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot 3\cdot \dots \cdot (n-1)\cdot n}{1\cdot 2\cdot 3\cdot \dots \cdot (n-1)\cdot n\cdot (n+1)}$$

Therefore, cancelling will result in everything on top cancelling but only the n+1 left on the bottom. So the answer is $\frac{1}{n+1}$

d. For the last one, we first need to make sense out of the expressions 2n – 1 and 2n + 1. It might be helpful to plug in a few values for n to see what these numbers are and how they relate to each other.

So for n = 1 we have 2n – 1 = 1 and 2n + 1 = 3.
For n = 2 we have 2n – 1 = 3 and 2n + 1 = 5.
For n = 3 we have 2n – 1 = 5 and 2n + 1 = 7.

We can see that these two expressions will always produce consecutive odd numbers. Therefore, if we write out the factorials, we know that every value smaller than or equal to the number 2n – 1 will cancel. We will be left with just the number after 2n – 1 (which is 2n) and the number 2n + 1.

So we have

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1}{2n(2n+1)}$$

Now that we have a good idea what a sequence is, we need to take the next step and talk about a series.

> **Definition**: **Series**- the sum of the first n terms of a sequence. We write
>
> $$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$
>
> where *i* is called the index of summation, *n* is the upper bound of summation and 1 is the lower limit of summation.

So, a series is basically what we get when we add up all of the terms of a finite sequence.

The notation used to describe a series is commonly called sigma notation (named for the Greek letter sigma, $\sum$, used in its notation).

As it turns out, series have a number of properties that are very useful in evaluating the sum that the series represents. Here is just a few such properties. A more extensive list can be found in all calculus textbooks.

Properties of Sums

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} 1 = n$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Example 4:

Find the sum.

a. $\sum_{i=1}^{4} 2i$ b. $\sum_{n=2}^{6}(1 - n^2)$ c. $\sum_{k=1}^{5} \frac{1}{(k-1)!}$ d. $\sum_{i=0}^{3} ix^i$

Solution:

a. The simplest way to figure out a sum when the upper bound is relatively small is by just writing the values out and preforming the sum. We get

$$\begin{aligned}\sum_{i=1}^{4} 2i &= 2(1) + 2(2) + 2(3) + 2(4)\\ &= 2 + 4 + 6 + 8\\ &= 20\end{aligned}$$

b. Again, we will just write out the values and add.

$$\sum_{n=2}^{6}(1-n^2) = (1-2^2)+(1-3^2)+(1-4^2)+(1-5^2)+(1-6^2)$$
$$= (-3)+(-8)+(-15)+(-24)+(-35)$$
$$= -85$$

c. Writing it out we get

$$\sum_{k=1}^{5}\frac{1}{(k-1)!} = \frac{1}{(1-1)!}+\frac{1}{(2-1)!}+\frac{1}{(3-1)!}+\frac{1}{(4-1)!}+\frac{1}{(5-1)!}$$
$$= \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}$$
$$= \frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}$$
$$= \frac{326}{120}$$
$$= \frac{163}{60}$$

d. Lastly, notice we have a variable in this sum. It makes no difference to us. We just simplify as much as possible.
We have

$$\sum_{i=0}^{3} ix^i = 0x^0+1x^1+2x^2+3x^3$$
$$= x+2x^2+3x^3$$

Even though we chose to just write out the values and add to evaluate the sums in example 4, we could have also used the properties of sums to do much of the same work. As it turns out, the properties are more useful with the upper bound of the sum is a very large number. We will leave a detailed discussion of that for another course.

15.1 Exercises

Find the first 4 terms of the sequence.

1. $a_n = n+1$
2. $a_n = n-2$
3. $a_n = 3n-4$
4. $a_n = 2-4n$
5. $a_n = \frac{(-1)^n}{3n}$
6. $a_n = (-1)^{n+1}\cdot 2^n$
7. $a_n = 1+(-1)^n$
8. $a_n = \frac{(-1)^n}{n-3}$
9. $a_1 = 2,\ a_{k+1} = 4a_k - 3$
10. $a_1 = 1,\ a_{k+1} = {a_k}^2$

Find the n-th term of the sequence.

11. 2, 3, 4, 5, …
12. 1, 4, 7, 10, …
13. 1, 4, 9, 16, …
14. 1, 8, 27, 64, …
15. 3, 5, 7, 9, …
16. 3, 7, 11, 15, …
17. 2, 4, 8, 16, …
18. $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\cdots$
19. $2,\sqrt{2}+1,\sqrt{3}+1,3,\cdots$

20. 2, -4, 6, -8, …

21. 1, 1, 2, 3, 5, …

22. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \cdots$

23. $1+\frac{1}{1}, 1+\frac{1}{2}, 1+\frac{1}{3}, 1+\frac{1}{4}, \cdots$

24. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots$

25. $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \frac{1}{8}, \cdots$

26. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \cdots$

Evaluate.

27. 4!

28. 6!

9. $\frac{3!}{5!}$

30. $\frac{4!}{8!}$

31. $\frac{8!}{2!\cdot 6!}$

32. $\frac{5!\cdot 4!}{6!}$

33. $\frac{2!\cdot 6!}{3!\cdot 5!}$

34. $\frac{3!\cdot 4!}{5!\cdot 6!}$

35. $\frac{n!}{(n-1)!}$

36. $\frac{(n+1)!}{(n-1)!}$

37. $\frac{2n!}{n!}$

38. $\frac{(n+2)!}{(n-1)!}$

39. $\frac{(2n+1)!}{(2n)!}$.

40. $\frac{(2n+2)!}{(2n+4)!}$

Find the sum.

41. $\sum_{i=1}^{5} 3i$

42. $\sum_{i=1}^{4} i^2$

43. $\sum_{i=1}^{5}(2i+1)$

44. $\sum_{k=1}^{3}(8k-3)$

45. $\sum_{n=1}^{5} \frac{3}{10^n}$

46. $\sum_{i=1}^{10} \frac{3}{i+1}$

47. $\sum_{k=3}^{6}(1+k^2)$

48. $\sum_{n=0}^{4} \frac{1}{n+1}$

49. $\sum_{i=0}^{4} 2^i$

50. $\sum_{i=0}^{4}(-3)^i$

51. $\sum_{k=2}^{5}(k+1)(k-3)$

52. $\sum_{k=2}^{6}(k^2-2k+1)$

53. $\sum_{n=0}^{8} \frac{1}{n!}$

54. $\sum_{k=0}^{4} \frac{(-1)^k}{k!}$

55. $\sum_{i=1}^{4}[(i-1)^2+(i+1)^3]$

56. $\sum_{n=0}^{5}[(n+1)-(n-1)]$

57. $\sum_{i=1}^{3} x^i$

58. $\sum_{k=0}^{4} x^{k+2}$

59. $\sum_{n=0}^{4} 2nx^{2n}$

60. $\sum_{i=1}^{5} i!\, x^i$