14.3 Graphing Polynomials

The last thing we would like to do with polynomials is talk about doing some basic graphing of polynomial functions. Unfortunately, to give a full treatment of graphing polynomials we would need to use Calculus. This section is intended, therefore, to just get the basics of graphing polynomials covered.

To graph polynomials we will need several “facts” to help us along the way. Many of these facts are easy to prove in a calculus course and, therefore, will not be proved here.

<u>Polynomial Graphs Fact 1</u>
Polynomials are continuous over all real numbers. That is, the graph has no “breaks” or corners (called a cusp). The graph is a smooth curve.

This is important because we have seen graphs which violate this fact. For example, the reciprocal function has a break, and the absolute value function has a cusp. Polynomials are nice, smooth, connected curves.

<u>Polynomial Graphs Fact 2</u>
A polynomial of degree *n* has at most *n – 1* turning points. A **<u>turning point</u>** is a point where a graph changes from rising to falling, or vice versa.

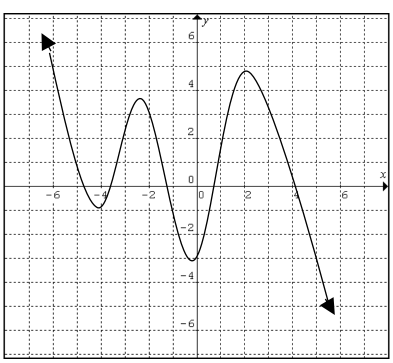
This fact will prove to be very helpful in making sure that we don’t make our graphs TOO curvy. So, a polynomial of degree 4, must have no more than 3 turning points in it, for example.

Be careful with this particular fact. This does **<u>not</u>** mean that a polynomial <u>must</u> have *n – 1* turning points. It just states that it can have **<u>no more</u>** than *n – 1* turning points. A polynomial may have less.

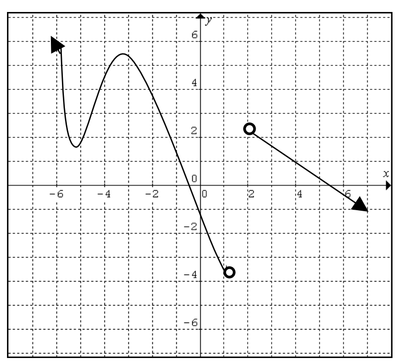
<u>Example 1</u>:

Explain why the graph is not the graph of $y = 4x^4 + 5x^3 - 1$.

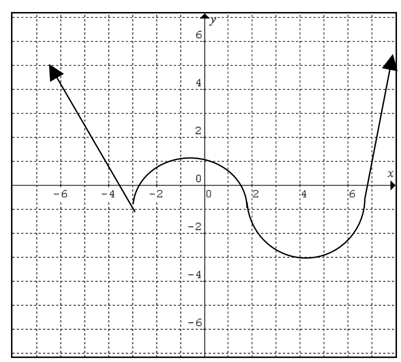
a.



b.



c.



Solution:

a. Since the degree of the polynomial is 4, the graph can have at most 3 turning point. Since this graph has 4 turning points, it cannot be the graph of the function.

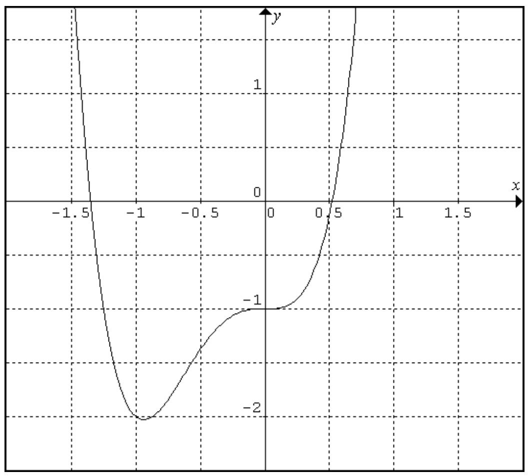
b. A polynomial function cannot have any “breaks.” This function has an obvious break between 1 and 2. Therefore, it cannot be the graph of the function.

c. Lastly, a polynomial cannot have any sharp corners, any cusp. This graph has a cusp at -3, so it cannot be the graph of the function.

So to get ourselves properly started with graphing, it is helpful to look at the graph of polynomial function. We can do this either by point plotting or use technology.

Let's take the actual graph from the above example. Again, this can be found by plotting points or technology.

The graph of $y = 4x^4 + 5x^3 - 1$ is



If we look at the graph, we can notice some behavior. We can see that the as we move out further and further along the positive and negative x-axis, the graph starts to resemble $y = 4x^4$.

The reason for this is the following. If we take and factor out the leading variable we would get

$$\begin{aligned} y &= 4x^4 + 5x^3 - 1 \\ &= x^4\left(4 + \frac{5}{x} - \frac{1}{x^4}\right) \end{aligned}$$

As x gets very large (we say as x approaches infinity and we write $x \to \infty$) the fraction values become very small (we say they approach 0 and we write $\frac{5}{x} \to 0$ and $\frac{1}{x^4} \to 0$).

So as $x \to \infty$

$$\begin{aligned} y &\approx x^4(4 + 0 - 0) \\ y &\approx 4x^4 \end{aligned}$$

This is where we get our next "fact" for graphing polynomials.

Polynomial Graphs Fact 3
For the graph of a polynomial function, as the *x* values get very large (either positive or negatively) the *y* values get large. Formally, as $x \to \pm\infty$, $y \to \pm\infty$. Furthermore, if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then as $x \to \pm\infty$, $f(x) \approx a_n x^n$.

More specifically,

- When the degree of the polynomial is **<u>even</u>** and
  - The leading coefficient is <u>positive</u>, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to \infty$.
  - The leading coefficient is <u>negative</u>, as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to -\infty$.

- When the degree of the polynomial is **<u>odd</u>** and
  - The leading coefficient is <u>positive</u>, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.
  - The leading coefficient is <u>negative</u>, as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.

These facts will simply help us with knowing the behavior of the graph as it continues off the rectangular grid.

Also, as it turns out, it also makes sense for us to use x- and y-intercepts as well as a sign chart to help us complete a graph of a polynomial function.

<u>Recall</u>:
- To find the x-intercepts, we let $y = 0$ and solve for x.
- To find the y-intercepts, we let $x = 0$ and solve for y.

Be aware. Now that we have numerous ways to solve and find roots of a polynomial, finding the x-intercepts could require us to use any number of the techniques that we have learned in this chapter so far.

Also, recall that a sign chart is used to find regions on which a polynomial (or any function) is entirely positive or entirely negative. What this means for graphing is regions where the polynomial is entire above the x-axis, or entirely below the x-axis.

For the sake of reference, the process of finding a sign chart is given below.

<u>Finding a Sign Chart of a Polynomial Function</u>
1. Find all roots of the polynomial function and plot them on the number line. 2. Use the number line from step 1 to determine the test intervals. 3. Test a value in each test interval from step 2 in the polynomial. This value will characterize the entire interval. 4. Complete the sign chart by filling in proper signs above the number line from step 1.

Now that we have all of this information, let's work on graphing a polynomial function.

<u>Example 2</u>:

Graph $f(x) = x^3 - 5x^2 - x + 5$ given that the turning points are (-0.1, 5.1) and (3.4, -17).

Solution:

The first thing we should do to graph a polynomial function is find all of the x- and y-intercepts. As we have always done, to get the x-intercepts we let the associated equation equal zero and solve. This polynomial factors by grouping. We get

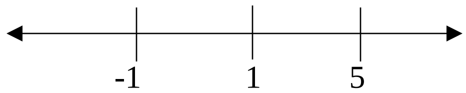
$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 \\ x^2(x-5) - (x-5) &= 0 \\ (x-5)(x^2-1) &= 0 \\ (x-5)(x-1)(x+1) &= 0 \end{aligned}$$

So clearly the x-intercepts are 5, 1 and -1. As we learned before, these will also be the roots that we use in creating our sign chart.

To find the y-intercept, we simply need to let x = 0 in the function and solve. It should be relatively obvious that we get 5.

The last thing we need in order to get all of the basic information that is required for graphing a polynomial is a sign chart. This will tell us when the polynomial is above the x-axis (positive) and below the x-axis (negative).

We start the sign chart by putting our x-intercepts on a number line, as follows:



Then, we test any point within each of the intervals to see if it results in a positive or a negative value. Using the factored form of the polynomial is usually the best for this.

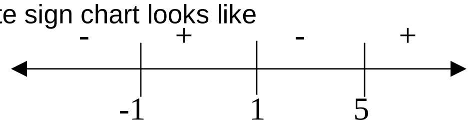
<u>Test -2</u>:
(-2-5)(-2-1)(-2+1)
$- \cdot - \cdot -$
$-$

<u>Test 0</u>:
(0-5)(0-1)(0+1)
$- \cdot - \cdot +$
$+$

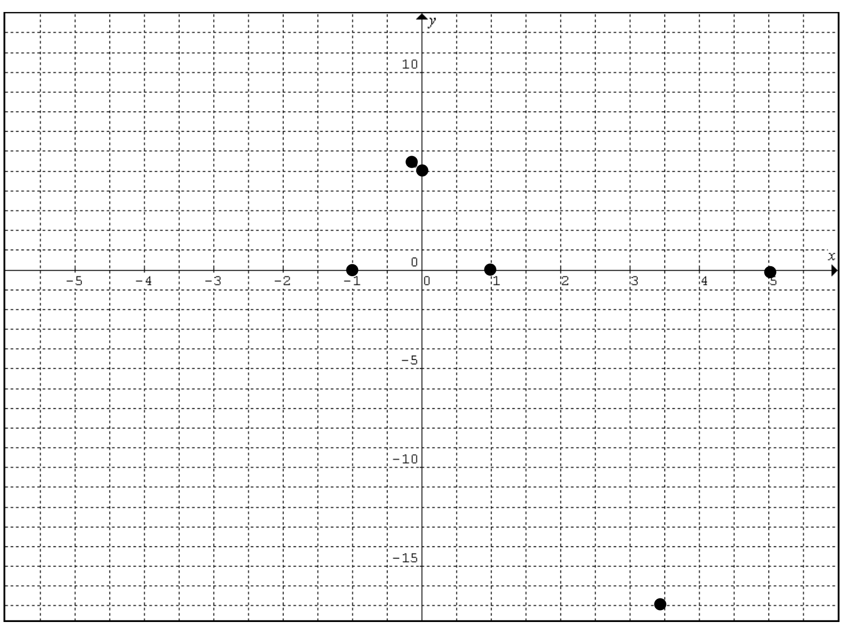
<u>Test 2</u>:
(2-5)(2-1)(2+1)
$- \cdot + \cdot +$
$-$

<u>Test 6</u>:
(6-5)(6-1)(6+1)
$+ \cdot + \cdot +$
$+$

So our complete sign chart looks like

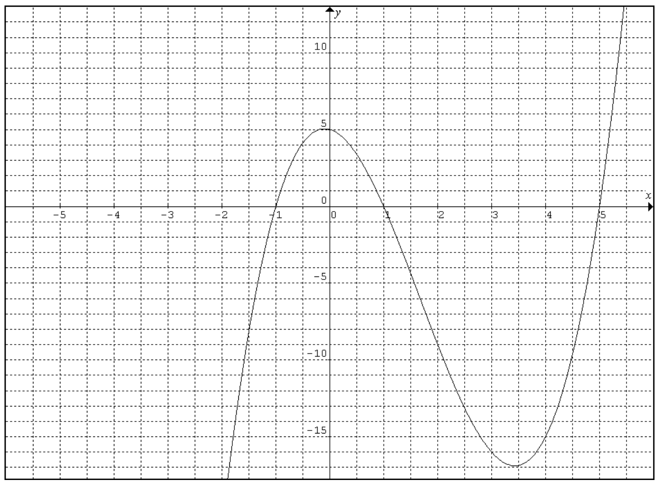


Now, we start by plotting the intercepts and our given turning points.



Then we use the sign chart and the fact that we have an odd degree with the leading coefficient that is positive (meaning as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$) to complete the graph.

We get



There is one other fact that we need to make our graphs as precise as we can for the time being. This fact will tell us how a graph acts near each of its x-intercepts.

Polynomial Graphs Fact 4
Let $f(x)$ be a polynomial and suppose $(x-a)^n$ is a factor of $f(x)$. Then, in the immediate vicinity of the x-intercept at a, the graph of $f(x)$ closely resembles $y = A(x-a)^n$.

What this means is that near each x-intercept, the graph looks like the power of the factor that is connected to that intercept.

Example 3:

Graph the following.

a. $y = -\frac{1}{4}(x-2)^3(x+1)^4$, turning point (0.7, 4.6)
b. $f(x) = x^3 - 4x^2 - 5x$, turning points (-0.5, 1.4) and (3.2, -24.2)
c. $g(x) = x^4 - x^3 - x^2 + x$, turning points (-0.6, -0.6) and (0.4, 0.2)

Solution:

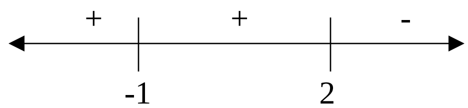
a. As we did in example 2, let's begin with finding the intercepts and constructing a sign chart.

   Obviously, the x-intercepts are -1 and 2, since the function is written in a factored form already.

   For the y-intercepts we have

$$y = -\frac{1}{4}(0-2)^3(0+1)^4$$
$$y = -\frac{1}{4} \cdot -8 \cdot 1$$
$$y = 2$$

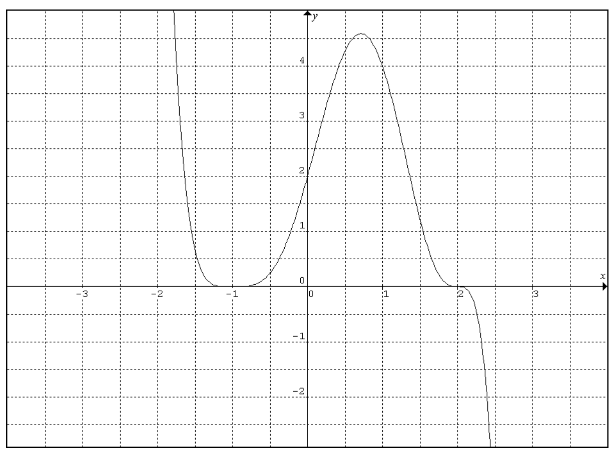
Using our x-intercepts to it is very simple to show that sign chart looks as follows (the details are left to the reader)



Although we now have a great deal of information, we need be careful when we produce the graph. Fact 4 above tells us that around the x-intercept of 2, the graph should closely resemble x^3. Also, around -1 the graph should resemble x^4, which is similar to x^2 but quite a bit more flat near the intercept.

The last detail that we need is to recognize that the degree is odd (since we add the powers together to obtain the degree, 3+7=7) and the leading coefficient is negative. So we know as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.

Putting all of this together, we get the graph



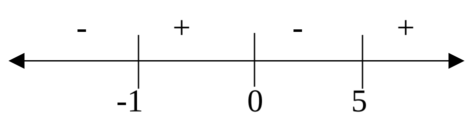
b. Its best to start by expressing the polynomial is a factored for, this way we can easily produce the x-intercepts and also it becomes much easier to generate the sign chart. If the polynomial does not factor, we would need to find the roots and use the linear factors theorem as we learned previously.

$$\begin{aligned} f(x) &= x^3 - 4x^2 - 5x \\ &= x(x^2 - 4x - 5) \\ &= x(x-5)(x+1) \end{aligned}$$

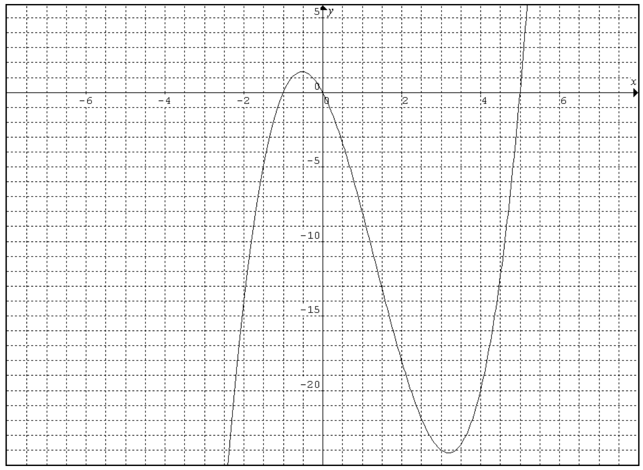
So our x-intercepts are -1, 0 and 5.

Its easiest here to find the y-intercept by substituting 0 into the original function. This gives us a y-intercept of 0.

Again, it can be shown that the sign chart is



Putting it all together with the turning points, and the behavior of an odd polynomial gives us our final graph.



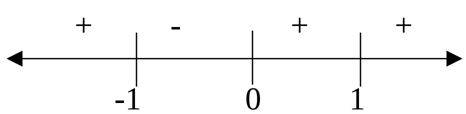
c. Finally, we begin by putting the function in a factored form.

$$\begin{aligned} g(x) &= x^4 - x^3 - x^2 + x \\ &= x(x^3 - x^2 - x + 1) \\ &= x\big(x^2(x-1) - 1(x-1)\big) \\ &= x(x-1)(x^2-1) \\ &= x(x-1)(x+1)(x-1) \\ &= x(x+1)(x-1)^2 \end{aligned}$$

So the x-intercepts are -1, 0 and 1.

The y-intercept (from the original function) is 0.

The sign chart, therefore would be



Using our sign chart, intercepts and turning points our graph is

<u>Note</u>: If we have not been given the turning points, the best approach at this point is to plot a point somewhere in the middle of the interval in which a turning point must exist. This will not result in a precise graph, but for the sake of a first course in algebra, the graph would be sufficient.

14.3 Exercises

Graph the following.

1. $y = (x-2)(x-1)(x+1)$, turning points: (-0.2, 2.1), (1.5, -0.6)
2. $y = (x+1)(x-2)(x+3)$, turning points: (-2.1, 4), (0.8, -8.2)
3. $f(x) = 2x(x+2)(x-1)$, turning points: (-1.2, 4.2), (0.5, -1.3)
4. $f(x) = x(x+5)(x-4)$, turning points: (-2.9, 42), (2.3, -29)
5. $g(x) = (x-3)(x+1)^3$, turning point: (2, -27)
6. $g(x) = (x-1)(x+2)^2$
7. $h(x) = -(x-2)^2(x+2)^2$
8. $h(x) = x^3(x-2)^2$, turning point: (1.2, 1.1)
9. $y = (x-1)^3(x+1)^4$, turning point: (0.1, -1.1)
10. $y = -x^3(x+1)^4$, turning point: (-0.4, 0.01)
11. $f(x) = x^3 - 2x^2 - 3x$, turning points: (-0.5, 0.9), (1.9, -6.1)
12. $f(x) = x^3 - x^2 + 4x$
13. $g(x) = -x^3 + 9x$, turning points: (-1.7, -10.4), (1.7, 10.4)
14. $g(x) = -x^3 + x^2$, turning point: (0.7, 0.1)
15. $h(x) = x^3 + 3x^2 - 4x - 12$, turning points: (-2.5, 1.1), (0.5, 13.1)
16. $h(x) = x^3 - 4x^2 - 25x + 100$, turning points: (-1.8, 126), (4.5, -2.4)
17. $g(x) = x^4 - 2x^3 - x^2 + 2x$, turning points: (-0.6, -1), (0.5, 0.6), (1.6, 1)
18. $g(x) = x^3 - 2x^2 - 4x + 8$, turning point: (-0.7, 9.5)
19. $f(x) = -2x^4 + 2x^2$, turning points: (-0.7, 0.5), (0.7, 0.5)
20. $f(x) = x^4 - 1$
21. $y = x^4 - 5x^2 + 4$, turning points: (-1.5, -2.3), (1.5, -2.3)
22. $y = x^4 + 3x^2 + 2$
23. $f(x) = x^5 + x^3 - 6x$, turning points: (-0.9, 4.1), (0.9, -4.1)
24. $f(x) = x^5 - x^4 - 20x^3$, turning points: (-3.1, 217), (3.9, -516)
25. $g(x) = 6x^3 + 14x^2 + x - 2$, turning point: (-1.5, 7.8)
26. $g(x) = 2x^3 + 3x^2 - 8x + 3$, turning points: (-1.8, 15.5), (0.8, -0.5)
27. $h(x) = 2x^4 + x^3 - 3x^2 - x + 1$, turning points: (-0.2, 1.1), (0.8, -0.4)
28. $h(x) = x^4 - 5x^3 + 3x^2 + x$, turning points: (-0.1, -0.1), (3.2, -25.1)
29. $y = -3x^4 - 5x^3 + 7x^2 + 12x + 4$, turning points: (-1.6, 3.5), (1, 15)
30. $y = -x^4 + x^3 - x^2 + 3x + 6$, turning point: (1,8)