14.2 Rational and Irrational Roots of Polynomials

In the previous section, we got a good idea of what roots are as well as some simpler examples of finding roots. However, we want to expand our ability to find roots well beyond the situations that we have had so far.

To do this, we will need several more theorems to guide us through this. For the sake of simplicity, we will give all of the theorems in this section without proof. The proofs of most of these can be found in the majority of Precalculus textbooks.

We start with the following.

The Rational Roots Theorem
Consider the polynomial equation $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$ where all the coefficients are integers. Let ${}^{p}/_{q}$ be a rational number where p and *q* have no common factors other than ± 1. If ${}^{p}/_{q}$ is a root of the equation, then *p* is a factor of a_0 and *q* is a factor of a_n.

The rational roots test is fairly easy to use to generate all the possible rational roots for a given polynomial function. Let's see an example.

Example 1:

List the possible rational roots of the following.

a. $9x^3 + 5x^2 - 17x - 8 = 0$

b. $18x^4 - x^3 + 12x^2 + 7x - 4 = 0$

Solution:

a. In order to find all the possible rational roots, we must use the rational root theorem. What the theorem tells us is we need all the factors of the leading coefficient as well as the factors of the constant term.

   The factors of the leading coefficient are called *q* and the factors of the constant term we call *p.* So we have

   Factors of *p*: 1, 2, 4, 8 Factors of *q*: 1, 3, 9

   So, to generate all the possible rational roots, we just need to generate all the ratios with these is such a way that the numbers have no common factors. Also, since we are generating potential roots, we need to be aware that each value could be positive or negative. So we attach a $\pm$ to each of these possible roots.

   So, taking each *p* value and putting it over each *q* value we get

$${}^{p}/_{q} = \pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm 4, \pm\frac{4}{3}, \pm\frac{4}{9}, \pm 8, \pm\frac{8}{3}, \pm\frac{8}{9}$$

   Therefore, these are all of the possible rational roots of the equation.

b. Again, we need to start by finding all of the factors for the leading coefficient, *q*, and constant term, *p*. We get

Factors of *p*: 1, 2, 4 Factors of *q*: 1, 2, 3, 6, 9, 18

So, now we generate our possible rational roots by placing each *p* over each *q* in such a way as to eliminate any that have a common factor.
This gives us

$$p/q = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{1}{9}, \pm\frac{1}{18}, \pm 2, \pm\frac{2}{3}, \pm\frac{2}{9}, \pm 4, \pm\frac{4}{3}, \pm\frac{4}{9}$$

Therefore, these are our possible rational roots.

The rational roots theorem is incredibly useful for finding roots. However, we need to be careful with it. It does not tell us what the roots of the polynomial function are, it simply gives us a list of all the possible rational roots.

In other words, the theorem is saying, "If the polynomial has rational roots, they must be in the list of roots from the rational roots test." That is to say, the function will not have any rational roots that do not appear in the list of roots from the rational roots test.

So, to find out what the ACTUAL rational roots for a polynomial are, we need to resort to the process of synthetic division that we learned in the last section. However, this can prove to be quite laborious. To help us cut down on the work, we have the following two theorems to use.

The Upper and Lower Bound Theorem

Consider the polynomial equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all the coefficients are real numbers and a_n is positive.

1. If we use synthetic division to divide $f(x)$ by *x – B*, where *B>0*, and we obtain a third row containing no negative numbers, then *B* is an upper bound for the real roots of $f(x) = 0$.

2. If we use synthetic division to divide $f(x)$ by *x – b*, where *b<0*, and we obtain a third row in which the numbers are alternately positive and negative, then *b* is an lower bound for the real roots of $f(x) = 0$.

The Location Theorem

If $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has at least one real root between *a* and *b*.

Let's see how all three of these work together to find roots of an equation.

Example 2:

Use the rational roots, upper and lower bound and location theorems to find the roots of the given equations.

a. $2x^3 - 5x^2 - 3x + 9 = 0$

b. $10x^4 + 107x^3 + 301x^2 + 171x + 23 = 0$

Solution:

a. The first thing we need to do is find all of the possible rational roots as we did in example 1. So we have

Factors of *p*: 1, 3, 9 Factors of *q*: 1, 2

So our possible rational roots are

$$ {}^{p}/_{q} = \pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}, \pm 9, \pm\frac{9}{2} $$

To figure out which of these potential roots are actual rational roots, we need to use the synthetic division process like we did in the last section. We learned that when synthetic division leaves a remainder of zero, then the value is a root.

We can start by picking any value that we would like in the list of possible rational roots. Lets start with something near the middle, say, 3. We get

3	2	-5	-3	9
		6	3	0
	2	1	0	9

Even though 3 is not a root, we can see that the third row contains no negative numbers. This means, according to the upper and lower bound theorem that 3 is an upper bound to the solutions of the equation. That tells us that none of the values that are larger than 3 could be a solution. Therefore we can throw out the 9 and 9/2 values as possible rational roots. So lets try something smaller, like 3/2.

$\frac{3}{2}$	2	-5	-3	9
		3	-3	-9
	2	-2	-6	0

So $\frac{3}{2}$ is a root to the equation. Not only that, we get a resulting equation $2x^2 - 2x - 6 = 0$. We can just use the quadratic formula to solve this equation to get the remainder of our roots. This gives

$$ \begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-6)}}{2(2)} \\ &= \frac{2 \pm \sqrt{52}}{4} \\ &= \frac{2 \pm 2\sqrt{13}}{4} \\ &= 1 \pm \sqrt{13} \end{aligned} $$

Therefore our solutions are $x = \frac{3}{2}, 1 \pm \sqrt{13}$

b. Again we need to begin by finding all the possible rational roots. We have

Factors of *p*: 1, 23 Factors of *q*: 1, 2, 5, 10

$$ {}^{p}/_{q} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}, \pm 23, \pm\frac{23}{2}, \pm\frac{23}{5}, \pm\frac{23}{10} $$

At first look this seems like an overwhelming list of possible values. However, once we start looking at the synthetic division table, we can immediately rule out every single positive value since there is no possible way the get a negative to appear in the last value of the second row to cancel out the 23.

This means that we need only concern ourselves with the negative values. Lets begin with a value that is more simple. It can be easily shown that -1 does not work.
So let's try – ½

$-\frac{1}{2}$	10	107	301	171	23
		-5	-51	-125	-23
	10	102	250	46	0

So, - ½ is a root. The first thing you should always do when you find a root is to check it for multiplicity. We can just use the third row of what we just to for this as well as for finding any other root. The reason for this is, every root of the resulting equation, must be a root of the original equation. Checking – ½ again will show that it is not a repeated root.

Let's try another value. Let's try $-\frac{1}{5}$

$-\frac{1}{5}$	10	102	250	46
		-2	-20	-46
	10	100	230	0

So, $-\frac{1}{5}$ is also a root and we have been left with a simple quadratic to solve for the remaining roots. It is left for the reader to solve $10x^2 + 100x + 230 = 0$ by using the quadratic formula. We get $x = -5 \pm \sqrt{2}$.

So the solutions are $x = -\frac{1}{2}, -\frac{1}{5}, -5 \pm \sqrt{2}$.

Although the process of finding roots this way is tedious, it does produce accurate solutions and is therefore quite valuable.

Also, as we see, not every root of a polynomial is going to be rational. So, we have another theorem which comes in very handy when finding roots.

The Conjugate Roots Theorem

Let $f(x)$ be a polynomial, all of whose coefficients are real numbers. Suppose that *a* + *bi* is a root of the equation $f(x) = 0$. Then *a* – *bi* is also a root of the equation.
Consequently, if $a + b\sqrt{c}$ is a root, then so is $a - b\sqrt{c}$.

So, this tells us that if we happen to know one complex or radical root of a polynomial equation, we get the conjugate for free since it also has to be a root.

Example 3:

Find all roots of the given equation.

a. $2x^3 + 11x^2 + 30x - 18 = 0$, *x* = -*3* – *3i* is a root
b. $x^6 - 2x^5 - 2x^4 + 2x^3 + 2x + 1 = 0$, $x = 1 + \sqrt{2}$ is a root

Solution:

a. According to the conjugate roots theorem, if -3 – 3i is a root of the equation, then -3 + 3i must also be a root. So, the first thing we need to do is divide out these two roots.

We have two primary ways that we can do this. We can either use synthetic division as we have been doing all along, or we can construct a quadratic that has these two values as solutions and then long divide out the quadratic.

As it turns out, most of the time it's easier to just use synthetic division. However, it will be a little more challenging and we must be more careful with the details, to do this properly. Lets start by dividing out the -3 – 3i. This will, of course, require us to remember how to multiply and combine complex numbers.

-3-3i	2	11	30	-18
		-6-6i	-33+3i	18
	2	5-6i	-3+3i	0

Even though our third row looks a little tough to deal with, it turns out to be just what we need when we divide through by -3 + 3i.

-3+3i	2	5-6i	-3+3i
		-6+6i	3-3i
	2	-1	0

This now just leaves us with the equation 2x – 1 = 0 to solve.
Clearly, the solution is x = ½ .

So, the solutions to the equation are $^1/_2, -3 \pm 3i$.

b. As is part a, since we know the root $1+\sqrt{2}$, we automatically know that $1-\sqrt{2}$ is also a root. So, we can divide out both of these roots using synthetic division, or, alternately, construct a quadratic with these values as roots and long divide.

Since synthetic division is easier we will stick with it. However, we need to be very careful with all of the details of the work. The good news is, you know that these values are roots, so if you do not get a remainder of 0, then you know that it was done incorrectly. We can do all the synthetic division at one time, as follows.

$1+\sqrt{2}$	1	-2	-2	2	0	2	1
		$1+\sqrt{2}$	1	$-1-\sqrt{2}$	-1	$-1-\sqrt{2}$	-1
$1-\sqrt{2}$	1	$-1+\sqrt{2}$	-1	$1-\sqrt{2}$	-1	$1-\sqrt{2}$	0
		$1-\sqrt{2}$	0	$-1+\sqrt{2}$	0	$-1+\sqrt{2}$	
	1	0	-1	0	-1	0	

Since we did the process of synthetic division twice, we need to reduce our degree of the resulting equation by 2. So this gives us the equation

$$x^4 - x^2 - 1 = 0$$

This equation now needs to be solved. As it turns out, the rational roots theorem will not provide any solutions to us. However, we can notice that the equation is in quadratic form. That is to say, we can solve it by making the substitution $u = x^2$.

This gives us

$$u^2 - u - 1 = 0$$

Since this quadratic does not factor, we need to use the quadratic formula to solve. We get

$$u = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

Resubstituting gives

$$x^2 = \frac{1+\sqrt{5}}{2} \qquad x^2 = \frac{1-\sqrt{5}}{2}$$
$$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}} \qquad x = \pm\sqrt{\frac{1-\sqrt{5}}{2}}$$

Finally, rationalizing the denominators gives us

$$x = \pm\frac{\sqrt{2+2\sqrt{5}}}{2} \qquad x = \pm\frac{\sqrt{2-2\sqrt{5}}}{2}$$

So the roots of the equation are $= 1 \pm \sqrt{2},\ \pm\frac{\sqrt{2+2\sqrt{5}}}{2},\ \pm\frac{\sqrt{2-2\sqrt{5}}}{2}$.

Finally, as it turns out, no that we know how to find all the roots of a polynomial function, we can express the function in a much more useful way with the following theorem.

The Linear Factors Theorem

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $f(x)$ can be expressed as a product of linear factors:

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

The complex numbers r_k are not necessarily distinct.

The Linear Factors theorem allows us to do two primary things. Write equations as a product of its linear factors, and generate equations from the roots of the equation.

Example 4:

Use the Linear Factors Theorem to express as a product of linear factors.

a. $f(x) = x^3 - 2x^2 - 3x$

b. $f(x) = x^3 + 2x^2 - 3x - 6$

Solution:

a. The Linear Factors theorem tells us that we should be able to express every function as a product of binomial, linear factors. To do this, all we need to do is find the roots of the polynomial and insert each of them into a binomial factor.

   The simplest way to do this is start by trying to factor the function, as this one does.

$$f(x) = x^3 - 2x^2 - 3x$$
$$= x(x^2 - 2x - 3)$$
$$= x(x-3)(x-1)$$

b. Again, we begin by trying to factor the function. This time we need to use the grouping technique since the polynomial has 4 terms.

$$
\begin{aligned}
f(x) &= x^3 + 2x^2 - 3x - 6 \\
&= x^2(x+2) - 3(x+2) \\
&= (x+2)(x^2-3)
\end{aligned}
$$

Since this does not factor any further, we need to solve the remaining nonlinear factors with other methods. In this case, we can see that $x^2 - 3 = 0$ would easily solve with extracting roots. This would give us $x = \pm\sqrt{3}$.

Therefore we can write the function as

$$
\begin{aligned}
f(x) &= (x+2)(x^2-3) \\
&= (x+2)(x-\sqrt{3})(x+\sqrt{3})
\end{aligned}
$$

<u>Example 5</u>:

Write a function that satisfies the given conditions.

a. Degree 3 with roots of $1 + i\sqrt{3}$ and 2
b. Degree 4 with roots of 5, $^3/_4$ and $3 - \sqrt{2}$

Solution:

a. Using the conjugate roots theorem, we know that we not only have the roots that are given, but $1 - i\sqrt{3}$ must also be a root.

   So, by the linear factors theorem, our function would be

$$
\begin{aligned}
f(x) &= \left(x - \left(1 + i\sqrt{3}\right)\right)\left(x - \left(1 - i\sqrt{3}\right)\right)(x-2) \\
&= \left(x - 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right)(x-2)
\end{aligned}
$$

   Now we need only to multiply all of these out and combine like terms.

$$
\begin{aligned}
f(x) &= \left(x - 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right)(x-2) \\
&= \left(x^2 - x + xi\sqrt{3} - x + 1 - i\sqrt{3} - xi\sqrt{3} + i\sqrt{3} + 9\right)(x-2) \\
&= (x^2 - 2x + 10)(x-2) \\
&= x^3 - 2x^2 - 2x^2 + 4x + 10x - 20 \\
&= x^3 - 4x^2 + 14x - 20
\end{aligned}
$$

b. As we saw above, we must also have a root of $3 + \sqrt{2}$. Also, to avoid have to use a fraction, we can choose to write the linear factor for ¾ as 4x – 3 since they have the same root. So this gives us the following. For the sake of brevity, we have consolidated many of the steps of multiplication and combining like terms.

$$
\begin{aligned}
f(x) &= \underbrace{(x-5)(4x-3)}\underbrace{\left(x - 3 - \sqrt{2}\right)\left(x - 3 + \sqrt{2}\right)} \\
&= (4x^2 - 23x + 15)(x^2 - 6x + 7) \\
&= 4x^4 - 47x^3 + 181x^2 - 251x + 105
\end{aligned}
$$

So, when it is all said and done, we can clearly see that the following theorem is true as an obvious consequence of the linear factors theorem.

The Polynomial Roots Theorem
Every polynomial of degree $n \geq 1$ has exactly *n* roots, where a root of multiplicity *k* is counted *k* times.

This is very helpful because if we are working with a 4th degree polynomial (for example) then we only need to find 4 roots.

14.2 Exercises

List the possible rational roots of the following.

1. $x^3 - 6x^2 + 5x - 6 = 0$
2. $x^3 - x^2 + 7x - 8 = 0$
3. $2x^3 + 3x^2 + 7x - 3 = 0$
4. $3x^3 + 7x - 2 = 0$
5. $4x^3 - 3x^2 - 1 = 0$
6. $6x^3 - 6x^2 + x + 5 = 0$
7. $2x^5 - 28x^3 + 16x - 25 = 0$
8. $6x^4 - 3x^3 + 2x^2 + x - 4 = 0$
9. $3x^4 - 7x^3 + 18x^2 + 5x - 18 = 0$
10. $5x^4 - 7x^3 - 7x^2 + 4x - 15 = 0$
11. $9x^4 - 3x^3 + 7x^2 - 5x + 12 = 0$
12. $3x^5 - 6x^3 + x - 10 = 0$

Find all roots of the given equation.

13. $2x^3 + 3x^2 - 8x + 3 = 0$
14. $3x^3 + 6x^2 - 2x - 4 = 0$
15. $2x^3 - x^2 + 6x - 3 = 0$
16. $x^3 - 5x^2 - 9x + 45 = 0$
17. $x^4 - x^3 - 5x^2 + 3x + 6 = 0$
18. $x^4 - 6x^2 - 27 = 0$
19. $x^5 + 4x^3 + 3x^2 = 5x^4 - 9x$
20. $2x^4 - x^3 - x^2 - 3x = 0$
21. $x^4 - 2x^3 - 4x^2 + 14x - 21 = 0$, $1 + i\sqrt{2}$ is a root
22. $2x^4 - 17x^3 + 137x^2 - 57x - 65 = 0$, $4 - 7i$ is a root
23. $x^4 - 4x^3 + 14x^2 - 36x + 45 = 0$, $2 + i$ is a root
24. $x^4 - 3x^3 - x^2 - 12x - 20 = 0$, $2i$ is a root
25. $x^5 - 9x^3 + 8x^2 - 8x + 8 = 0$, 1 and $\pm 2\sqrt{2}$ are roots
26. $x^5 - 5x^4 + 30x^3 + 18x^2 + 92x - 136 = 0$, $-1 + i\sqrt{3}$ and $3 + 5i$ are roots
27. $x^6 - 2x^5 - 3x^4 + 4x^3 + 3x^2 - 2x - 1 = 0$, $1 + \sqrt{2}$ is a root
28. $x^5 - 3x^4 + 6x^3 + 2x^2 - 60x = 0$, $1 + 3i$ is a root
29. $x^5 + x^3 + 2x^2 - 12x + 8 = 0$
30. $x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32 = 0$

Write a function that satisfies the given conditions.

31. Degree 3 with roots of 2 and $1 \pm \sqrt{3}$
32. Degree 3 with roots of -3 and $2 \pm \sqrt{2}$
33. Degree 4 with roots of -4, $\frac{1}{3}$ and $\pm\sqrt{6}$
34. Degree 4 with roots of 3, $\frac{1}{2}$ and $\pm 2i$
35. Degree 4 with roots of -1, $-\frac{2}{3}$ and $1 + \sqrt{5}$
36. Degree 4 with roots of $\frac{3}{4}$, $\frac{1}{2}$ and $\sqrt{2}$
37. Degree 5 with roots of $\frac{1}{2}$, 4, 0 and $3 - \sqrt{2}$
38. Degree 5 with roots of $\frac{3}{5}$, $1 - i\sqrt{3}$ and $-2 + \sqrt{5}$
39. Degree 6 with roots of i, 2i and 3i
40. Degree 6 with roots of 1 + i, 2 – i and $1 + \sqrt{2}$