14.1 Polynomial Functions

Now that we have a very broad view of many topics in a first year algebra course. We want to take an extended look at just a couple more topics in this chapter. Namely, we want to revisit the polynomials and rationals that we looked at in the first half of the text.

In particular, we want to turn these into functions and be able to do some basic graphing of them. We start with the polynomial.

> **<u>Definition</u>**: **<u>Polynomial Function</u>**- a function of the form
>
> $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
>
> where every a_i is a real number and *n* is an integer.

In other words, a polynomial function is simply a polynomial (like we learned in chapter 3) which is in function form.

> **<u>Definition</u>**: **<u>Root of a polynomial</u>**- also known as **<u>solution</u>** or **<u>zero</u>**- all values (real or complex) that keep the equation a true statement.

Roots of a polynomial are same thing as solutions to the polynomial equation. The word solution is the more general term for a value that properly satisfies an equation of any type. Roots, on the other hand, usually only refer to polynomial function solutions.

So, the basic idea is, if a value makes the polynomial function equal to zero, then it is a root of the function.

Let's solidify the idea with the following examples.

<u>Example 1</u>:

Is $\sqrt{2}$ a root of $g(x) = -x^5 + x^3 + 2x$?

Solution:

In order to be a root, the value must make the polynomial function zero when the function is evaluated at the given value. So, plugging in the $\sqrt{2}$ to the function gives us

$$\begin{aligned} g(\sqrt{2}) &= -\sqrt{2}^5 + \sqrt{2}^3 + 2\sqrt{2} \\ &= -4\sqrt{2} + 2\sqrt{2} + 2\sqrt{2} \\ &= 0 \end{aligned}$$

Since we got a zero, this means that $\sqrt{2}$ is, in fact, a root of the polynomial.

<u>Example 2</u>:

Find the roots of the following.

a. $f(x) = 2x^2 - 2x - 5$

b. $h(x) = x(x-2)(x-2)$

Solution:

a. In light of example 1, we see that the roots are simply the solutions to the equation we get when we set the function equal to zero. So we just need to solve

$$f(x) = 2x^2 - 2x - 5 = 0$$

Since this is just a simple, nonfactorable quadratic, we can solve with the quadratic formula. We get

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-5)}}{2(2)} \\ &= \frac{2 \pm \sqrt{4+40}}{4} \\ &= \frac{2 \pm \sqrt{44}}{4} \\ &= \frac{2 \pm 2\sqrt{11}}{4} \\ &= \frac{1 \pm \sqrt{11}}{2} \end{aligned}$$

Therefore the roots of the function are $\frac{1\pm\sqrt{11}}{2}$

b. Again, we simply need to solve the equation resulting from setting the function equal to zero. This time, however the function is clearly already factored for us. So we can just set each factor equal to zero. This gives us

$$\begin{array}{lll} x(x-2)(x-2) = 0 & & \\ x = 0 & x - 2 = 0 & x - 2 = 0 \\ & x = 2 & x = 2 \end{array}$$

So, the roots are 0 and 2.

As we saw in the last example, sometimes a particular root is more than just a root. Sometimes, we get a root that shows up more than one time. We call these repeated roots.

Definition: **Repeated Root**- a root that appears more than one time when listing roots individually. If a root is repeated twice, it is called a **double root**. If a root is repeated *k* times, we call it a **root of multiplicity k**.

Example 3:

List the roots with their multiplicity.

a. $(x-1)(x+1)^2 = 0$ b. $x^3(x-3)^2(x+7)^4 = 0$ c. $x^3 - 8x^2 + 16x = 0$

Solution:

a. Finding the roots is as simple as solving the equation. The multiplicity can be determined by the power associated with the factor from which the root comes. So we begin by solving the equation.

$$\begin{array}{ll} (x-1)(x+1)^2 = 0 & \\ x - 1 = 0 & (x+1)^2 = 0 \\ x = 1 & x = -1 \end{array}$$

Since the factor from the root 1 has no power, its multiplicity is 1

However, since the factor from the root -1 has a squared, its multiplicity is 2.

b. In light of part a, we can clearly see we have roots of 0, 3 and -7.
Attaching the powers to the corresponding roots we get the following solution

0 with multiplicity 3, 3 with multiplicity 2 and -7 with multiplicity 4

c. In this last part, we need to do some preliminary work with the equation to determine its roots and their multiplicity.

$$x^3 - 8x^2 + 16x = 0$$
$$x(x^2 - 8x + 16) = 0$$
$$x(x-4)(x-4) = 0$$
$$x(x-4)^2 = 0$$

So clearly we have 0 with multiplicity of 1 and 4 with multiplicity of 2

Now that we have a decent handle on the idea of finding roots. We want a more simple way to determining if a value is a root or not. The following two theorems will help us with this.

The Remainder Theorem
When a polynomial $f(x)$ is divided by $x - r$, the remainder is $f(r)$.

The Factor Theorem
Let $f(x)$ be a polynomial. $f(r) = 0$ if and only if $x - r$ is a factor of $f(x)$.

We omit the proof of these theorems here, however, they both follow very simply from the division algorithm. You can find the proof in a pre-calculus text. For the sake of this course, it is more important that we know what the implications of these two theorems are.

In a nutshell, what these tell us is…

If you divide a polynomial by *x – r* and you get a remainder of 0, then *r* is a root of the polynomial.

This proves to be very helpful in finding roots (or solutions) to polynomials. We can rely heavily on synthetic division to get us through this as we see in the next examples.

Example 4:

Use the remainder theorem to evaluate the given polynomial at the given value.

a. $f(x) = 2x^2 - x - 4 \; at \; x = 4$

b. $f(x) = x^5 - x^4 - x^3 - x^2 - x - 1 \; at \; x = -2$

c. $f(x) = x^7 - 7x^6 + 5x^4 + 1 \; at \; x = -3$

Solution:

a. According to a straight forward application of the remainder theorem, to evaluate the function at 4, we just need to divide out *x – 4* and determine the remainder.

The simplest way to do this is to use synthetic division to do so. Recall, that when we use synthetic division, the very last number we get, is the remainder. Therefore, we will divide, synthetically, and our remainder will be $f(4)$. We proceed as follows

4	2	-1	-4
		8	28
	2	7	24

Since the reminder is 24, then $f(4) = 24$

b. As we did in part a, we just need to synthetically divide by -2 and find the remainder. We get

-2	1	-1	-1	-1	-1	-1
		-2	6	-10	22	-42
	1	-3	5	-11	21	-43

So $f(-2) = -43$

c. Finally, we will use synthetic division as we did in parts a and b.

-3	1	-7	0	5	0	0	0	1
		-3	30	-90	255	-765	2295	-6885
	1	-10	30	-85	255	-765	2295	-6884

So $f(-3) = -6884$

Example 5:

Solve using the given root.

a. $x^3 + 7x^2 + 11x + 5 = 0, \quad -1 \textit{ is a root}$

b. $2x^3 - 5x^2 - 46x + 24 = 0, \quad 6 \textit{ is a root}$

c. $6x^5 - 19x^4 - 25x^3 + 18x^2 + 8x = 0, \quad 4 \textit{ and } -\frac{1}{3} \textit{ are roots}$

Solution:

a. By the factor and remainder theorems we know that since -1 is given as a root, *x+1* must be a factor of the polynomial. Therefore we can use synthetic division to divide it out and leave us with a polynomial that we can solve with other methods. This gives

-1	1	7	11	5
		-1	-6	-5
	1	6	5	0

We already knew that the remainder would be 0 by the remainder and factor theorems. But now, like we learned in synthetic division, we also have the resulting equation from when we divided out the root of -1. We have

$$x^2 + 6x + 5 = 0$$

Now we just need to solve this equation to get the remaining roots. This quadratic clearly factors and so we will just solve it by factoring.

$$x^2 + 6x + 5 = 0$$
$$(x + 1)(x + 5) = 0$$
$$x = -1 \; and \; x = -5$$

So the roots, or solution set is {-1 (with multiplicity 2), -5}.

b. As in part a above, we will start with dividing out the given root.

6	2	-5	-46	24
		12	42	-24
	2	7	-4	0

This leaves us with the resulting quadratic $2x^2 + 7x - 4$ which we can again easily solve with factoring.

$$2x^2 + 7x - 4 = 0$$
$$(2x - 1)(x + 4) = 0$$
$$x = \frac{1}{2} \; and \; x = -4$$

So our solutions set is {6, ½, -4}.

c. Lastly, we have two roots given to us. But before we start synthetically dividing those out, we need to notice that the polynomial has a GCF of *x*. This needs to be factored out first

$$6x^5 - 19x^4 - 25x^3 + 18x^2 + 8x = 0$$
$$x(6x^4 - 19x^3 - 25x^2 + 18x + 8) = 0$$

This provide us with another root, 0. Now we can use what we learned in parts a and b to get out the remaining two given roots and hopefully leave us with an equation that we can solve. We get

4	6	-19	-25	18	8
		24	20	-20	-8
	6	5	-5	-2	0

We can now just continue by using the bottom row without converting it into a polynomial first

$-\frac{1}{3}$	6	5	-5	-2
		-2	-1	2
	6	3	-6	0

Now we only have the quadratic equation $6x^2 + 3x - 6 = 0$ left to solve. Once we factor out the GCF of 3 we have

$$3(2x^2 + x - 2) = 0$$

Since this does not solve by factoring, we will have to use the quadratic formula. This gives

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)}$$
$$= \frac{-1 \pm \sqrt{1 + 16}}{4}$$
$$= \frac{-1 \pm \sqrt{17}}{4}$$

So the solutions set is { 0, 4, $-\frac{1}{3}$, $\frac{-1\pm\sqrt{17}}{4}$}.

14.1 Exercises

Is the given value a root to the given function?

1. $f(x) = 2x^2 - 3x + 1.\ x = \frac{1}{2}$
2. $g(x) = x^2 + x - 1.\ x = 1$
3. $h(x) = 5x^3 + x^2 + 2x + 8.\ x = -1$
4. $h(x) = x^3 + x^2 - 7x + 5.\ x = 1$
5. $f(x) = -2x^5 + 3x^4 + 8x^3.\ x = 0$
6. $f(x) = 5x^4 - 3x^3 + 7x^2 + x.\ x = -1$
7. $g(x) = (x-2)(x+3)(x-7),\ x = -2$
8. $g(x) = (2x+3)(x-4)(x+1).\ x = -\frac{3}{2}$
9. $h(x) = x^4 + 8x^3 + 9x^2 - 8x - 10.\ x = \sqrt{6} - 4$
10. $h(x) = x^4 + 8x^3 + 9x^2 - 8x - 10.\ x = \sqrt{6} + 4$

Find the roots and list the multiplicity.

11. $(x-2)(x+1) = 0$
12. $(x-5)(x+5) = 0$
13. $(x-3)(x+4)^2 = 0$
14. $(x+1)^2(x-2) = 0$
15. $\frac{1}{2}(x-1)^3(x+2)^2 = 0$
16. $\frac{2}{3}(x+6)^2(x+7)^4 = 0$
17. $(3x+7)^3(x+3)^3 = 0$
18. $(x-1)^3(2x+5)^2 = 0$
19. $x^3(6x-1)^2(x-4)^4 = 0$
20. $2x(x-1)(x+1)^5(x-2)^2 = 0$
21. $4x^2 + 4x + 1 = 0$
22. $x^2 + 6x + 9 = 0$
23. $x^4 + 2x^3 + x^2 = 0$
24. $x^5 - 4x^4 + 4x^3 = 0$
25. $x^3 - 2x^2 - 4x + 8 = 0$
26. $x^4 + 3x^3 - 9x^2 - 27x = 0$

Solve using the given root.

27. $x^3 - 2x^2 - x + 2 = 0$, 2 is a root
28. $x^3 - 4x^2 - 9x + 36 = 0$, -3 is a root
29. $x^3 - 2x^2 - 5x + 6 = 0$, 3 is a root
30. $x^3 + 7x^2 + 11x + 5 = 0$, -1 is a root
31. $x^3 + x^2 - 7x + 5 = 0$, 1 is a root
32. $x^3 + x^2 - x + 2 = 0$, -2 is a root
33. $3x^3 - 10x^2 + x + 6 = 0$, -2/3 is a root
34. $2x^3 + x^2 - 5x - 3 = 0$, -3/2 is a root
35. $3x^3 - 7x^2 + 8x - 2 = 0$, 1/3 is a root
36. $x^3 + 3x^2 + 5x + 6 = 0$, -2 is a root
37. $9x^4 - 9x^3 - 19x^2 + x + 2 = 0$, 2 and 1/3 are roots
38. $x^4 - 2x^3 + 11x - 10 = 0$, -2 and 1 are roots
39. $x^4 - 15x^3 + 75x^2 - 125x = 0$, 5 is a root
40. $3x^4 - 5x^3 - 16x^2 + 12x = 0$, -2 is a root
41. $x^4 + 2x^3 - 23x^2 - 24x + 144 = 0$, -4 and 3 are roots
42. $6x^4 - 19x^3 + 52x^2 - 29x - 30 = 0$, 5 and 3 are roots
43. $6x^4 - 19x^3 - 25x^2 + 18x + 8 = 0$, 4 and -1/3 are roots
44. $2x^4 - 21x^3 + 63x^2 - 34x - 30 = 0$, 5 and 3/2 are roots
45. $6x^4 - 17x^3 + 25x^2 + 11x - 10 = 0$, 1/2 and -2/3 are roots
46. $6x^4 - 37x^3 + 66x^2 - 26x - 15 = 0$, 3/2 and 5/3 are roots
47. $x^5 - 2x^4 - 11x^3 + 18x^2 + 30x - 36 = 0$, -2, 3 and 1 are roots
48. $4x^5 - 15x^4 + 8x^3 + 19x^2 - 12x - 4 = 0$, 2, 1 and -1 are roots
49. $2x^5 - x^4 - 6x^3 + 3x^2 - 8x + 4 = 0$, -2, 2, and 1/2 are roots
50. $2x^5 - 5x^4 + 7x^3 + 14x^2 - 54x + 36 = 0$, -2, 1 and 3/2 are roots