13.5 Inequalities of Two Variables

Now that we have taken a look at systems of equations, we want to work our way towards dealing with a systems of inequalities.

The first thing we need to do is refresh our memories about linear inequalities as well as take a look at other inequalities of two variables.

> **Definition**: **Linear Inequality of Two Variables**- A linear equation of two variables with an inequality symbol instead of an equal sign.

As usual, we are interested in capturing all the solutions which we do so by graphing.

Recall the method we used to graph.

Graphing a Linear Inequality in Two Variables
1. Replace the inequality symbol with an equal sign 2. Graph the equation (Dotted for < and >; Solid for $\leq$ and $\geq$) 3. Test a point not on the line. If the point satisfies the inequality, shade the portion containing the point, if not, shade the other portion.

Recall, the reason for the dotted line for < and > verses the solid line for ≤ and ≥ is because with the < or > symbol, the equation isn't an included part of the solutions. However, for the ≤ and ≥, the equation is actually part of the solution since they have an "or equal to" component.

Example 1:

Graph the given inequality.

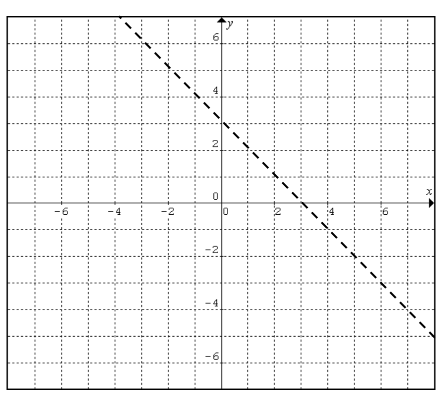
a. $x + y < 3$ b. $x - 3y < 6$

Solution:

a. To graph the inequality, the first thing we do is change $x + y < 3$ to $x + y = 3$. Now, we graph the equation the same way we always did, the "slope-intercept-intercept" method.

It can be easily shown that we get m = -1, y-intercept of (0, 3) and x-intercept of (3, 0).

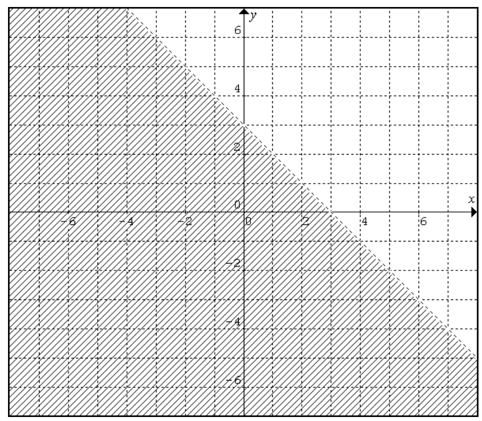
Now we draw the graph with a dotted line, since our inequality symbol is <.



Testing (0,0) for simplicity gives

$$x + y < 3$$
$$0 + 0 < 3$$
$$0 < 3$$

So since we got a true statement, (0, 0) must be a solution. Therefore, every ordered pair in the region containing (0, 0) is a solution. So we shade the side of the line containing (0, 0) to get our completed graph.



b. Again start by changing $x - 3y < 6$ to $x - 3y = 6$ and then graphing.

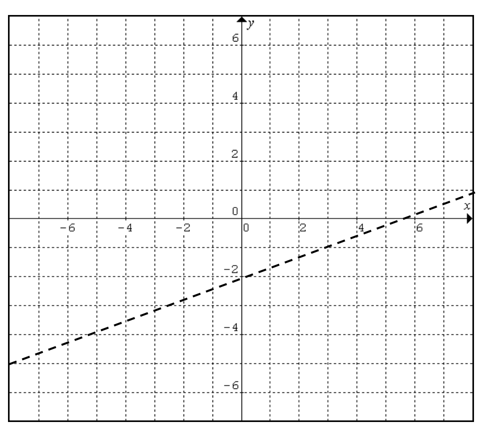
Slope-intercept:

$$x - 3y = 6$$
$$-x \qquad -x$$
$$\frac{-3y}{-3} = \frac{-x}{-3} + \frac{6}{-3}$$
$$y = \frac{1}{3}x - 2$$

x-intercept:

$$x - 3y = 6$$
$$x - 3(0) = 6$$
$$x = 6$$

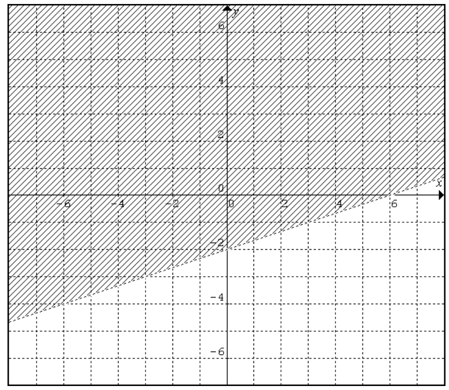
So m = $\frac{1}{3}$, y-intercept is (0, -2) and x-intercept is (6, 0). We graph with a dotted line because we are dealing with <.



Now we test (0, 0) to get

$$x - 3y < 6$$
$$0 - 3(0) < 6$$
$$0 < 6$$

So we shade the side containing (0, 0)



Since we have looked at far more than just linear equations, we want to expand this idea to inequalities of many other types that contain two variables.

Fortunately, we can use the exact same method that we just used for linear inequalities. We will graph the associated curve, then choose a test point to determine the shading.

Example 2:

Graph the given inequality.

a. $y > 2x^2 - 8x + 3$ b. $y \geq \frac{2}{x-1} + 2$ c. $\frac{(x-2)^2}{4} + \frac{(y+4)^2}{9} > 1$

Solution:

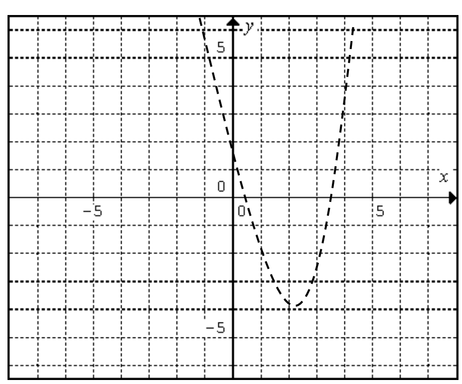
a. As we did with the linear type, we start by graphing $y = 2x^2 - 8x + 3$ but using a dotted line because of the > symbol. Recall, to graph we need to find the vertex and intercepts. Completing square to get standard form gives us

$$y = 2(x-2)^2 - 5$$

So our vertex is (2,5).

Solving for our intercepts gives x-intercepts of $\left(2 \pm \frac{\sqrt{10}}{2}, 0\right)$ or (0.4,0) and (3.6,0). The y-intercepts is clearly (0,3).

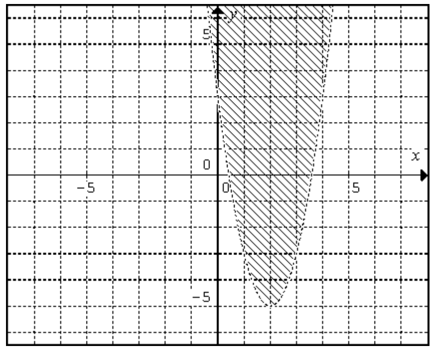
So we get



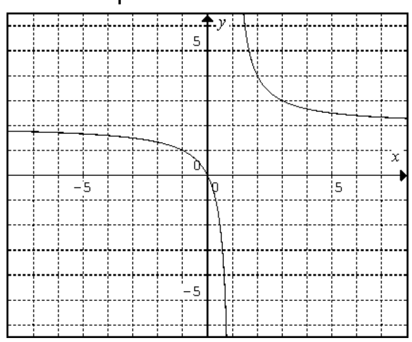
Now we just need to determine the shading. As we did before, it is simplest to test (0,0). We get

$$0 > 2(0)^2 - 8(0) + 3$$
$$0 > 3$$

Since that is false, we shade the part that does not contain the point (0,0), which means we shade the inside of the parabola. We get



b. Again, we start by graphing $y = \frac{2}{x-1} + 2$ with a solid curve since the equation is included this time.
The graph here is simply the reciprocal function which is stretched by a factor of 2, shifted right 1 and shifted twice up.



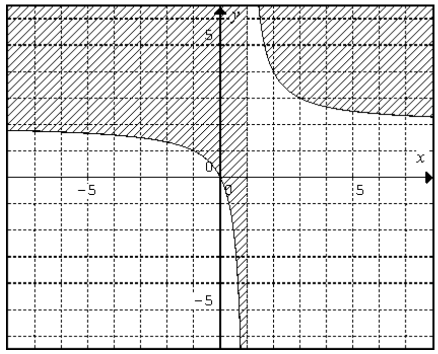
In this case, testing a point causes a couple of problems. First, testing (0,0) is not possible this time since the graph goes through (0,0). So instead we use a different point, say (2,0). Normally, (1,0) would be a suitable alternate, but 1 is not is the domain and therefore cannot be used.
We get

$$0 \geq \frac{2}{2-1} + 2$$
$$0 \geq 2 + 2$$
$$0 \geq 4$$

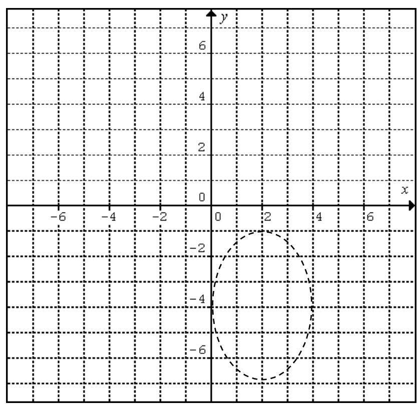
Since this is false, we know that the region below the right branch is not shaded. However, there is another problem. This test only gives us information about the branch that is above or below the point we are testing. So, we need to run another test for the left branch.
We can use (-1,0) since it is directly above or below the left branch. We get

$$0 \geq \frac{2}{-1-1} + 2$$
$$0 \geq -1 + 2$$
$$0 \geq 1$$

So we need to shade above the left branch as well. We will need to use this “double” testing method anytime our graph has breaks in its domain.
So we have,



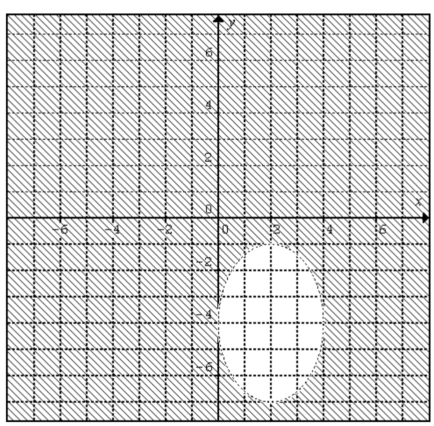
c. Lastly, we start by graphing the ellipse $\frac{(x-2)^2}{4} + \frac{(y+4)^2}{9} = 1$ with a dotted curve since we are working with >. Clearly, the ellipse has center (2,-4) and major axis of length 6 (vertical) and minor axis of length 4 (horizontal).
So we get



Now testing (0,0) we get

$$\frac{(0-2)^2}{4} + \frac{(0+4)^2}{9} > 1$$
$$1 + \frac{16}{9} > 1$$
$$\frac{25}{9} > 1$$

So we shade the part outside the ellipse since the test point resulted in a true statement.



13.5 Exercises

Graph the given inequality.

1. $y > -\frac{2}{3}x + 1$
2. $y \le -\frac{3}{5}x - 2$
3. $3x + 2y < 12$
4. $2x - 5y \ge 15$
5. $3x - 4y > -8$
6. $3x > 2 - y$
7. $2x < 3y - 6$
8. $x - 3y > 3$
9. $2x - 5y > 0$
10. $3x + y \le 0$
11. $y < x^2 + 1$
12. $y \ge x^2 + x$
13. $y \ge 3x^2 + 6x - 1$
14. $y < 2x^2 + 4x + 3$
15. $x + y^2 < 4$
16. $x^2 + y^2 > 9$
17. $x^2 + y^2 \le 4$
18. $x - y^2 > 4$
19. $y < \sqrt{x} - 2$
20. $y \le \sqrt{x+1} - 2$
21. $y < -|x| + 2$
22. $y \ge |x+1| + 2$
23. $y < -(x+2)^3 - 2$
24. $y < (x-1)^3 + 2$
25. $y \le 3^{x+1}$
26. $y \le 4^x - 2$
27. $y > 4^{-x} - 1$
28. $y > -2^{-x} - 3$
29. $y \ge -log_3(x-1)$
30. $y > log_4 x - 2$
31. $y < 2 + log_5(x+2)$
32. $y \ge -log_3(-x)$
33. $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} < 1$
34. $\frac{(x+2)^2}{16} + \frac{(y-2)^2}{4} \le 1$
35. $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} \ge 1$
36. $(y+1)^2 - \frac{(x+1)^2}{4} > 1$
37. $y^2 - 2y + 4x - 3 < 0$
38. $y^2 - 2y - x - 5 \ge 0$
39. $x^2 + 4y^2 + 6x - 8y + 9 < 0$
40. $y^2 - 4x^2 + 24x + 4y - 41 > 0$