

13.4 Nonlinear Systems

In this final section, we want to learn how to solve systems of equations that are not necessarily all linear. We call these non-linear systems of equations.

Definition: Non-linear system of equations

A system of equations where one or more equations involved is not a line.

We primarily use the substitution method to solve a non-linear system. However, sometimes elimination will work as well.

Also, just as before, the solution to a non-linear system is all the points of intersection of the graphs of the equations. Therefore, since we now have more than just lines, we can have a variety of numbers of solutions. The graph can intersect once, twice, several times or not at all. Therefore, we should always verify the solution(s) to a system by looking at the graph.

Example 1:

Solve the system.

a. $x^2 + y^2 = 100$
 $y - x = 2$

b. $x^2 + y^2 = 25$
 $y^2 = x + 5$

c. $x^2 + 2y^2 = 12$
 $xy = 4$

Solution:

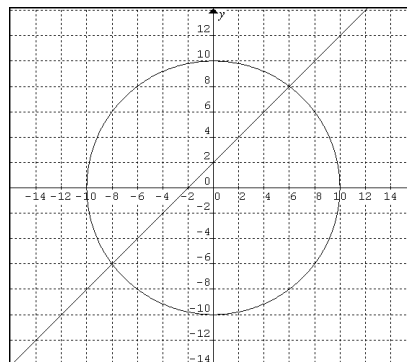
- a. We will solve this system by substitution. So we start by solving the bottom equation for y and then substitute it into the top equation. We get

$$\begin{aligned}y - x = 2 &\Rightarrow y = x + 2 \\x^2 + y^2 = 100 &\Rightarrow x^2 + (x + 2)^2 = 100 \\x^2 + x^2 + 4x + 4 &= 100 \\2x^2 + 4x - 96 &= 0 \\2(x^2 + 2x - 48) &= 0 \\2(x + 8)(x - 6) &= 0 \\x &= -8, 6\end{aligned}$$

So we have two different x values. This means we should have two points of intersection. Let's now find the y values and verify with a graph.

$$\begin{aligned}y &= x + 2 & y &= x + 2 \\y &= -8 + 2 & y &= 6 + 2 \\y &= -6 & y &= 8\end{aligned}$$

So our solutions are $(-8, -6)$ and $(6, 8)$. We clearly have a circle and a line, thus we can easily graph them together and get



- b. Again we will use substitution to solve. This time notice that the bottom equation is already solved for y^2 and we have a y^2 in the top equation. Thus, that is the substitution we will make. We get

$$\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= x + 5 \end{aligned} \Rightarrow x^2 + (x + 5) = 25$$

Now we solve for x and then solve for y .

$$x^2 + (x + 5) = 25$$

$$x^2 + x - 20 = 0$$

$$(x + 5)(x - 4) = 0$$

$$x = -5, 4$$

We substitute these back in to get y . We get

$$y^2 = -5 + 5$$

$$y^2 = 4 + 5$$

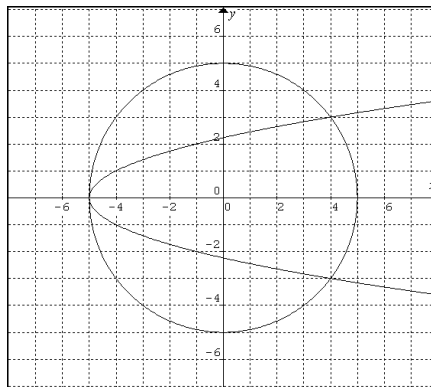
$$y^2 = 0$$

$$y^2 = 9$$

$$y = 0$$

$$y = \pm 3$$

So we have three different solutions $(-5, 0)$, $(4, 3)$ and $(4, -3)$. Lets verify with a graph. We have here a parabola and a circle. We get



- c. Again, we will use substitution to solve. We need to decide which variable to solve for first. It seems that x or y on the bottom equation would be easiest. So we will solve for x . We get

$$xy = 4 \Rightarrow x = \frac{4}{y}$$

Now we substitute that into the first equation and solve. We have

$$\begin{aligned} x^2 + 2y^2 &= 12 \Rightarrow \left(\frac{4}{y}\right)^2 + 2y^2 = 12 \\ \frac{16}{y^2} + 2y^2 &= 12 \end{aligned}$$

We will have to clear the fractions and solve as we did in chapter 10, that is, using a substitution.

$$y^2 \left(\frac{16}{y^2} + 2y^2 \right) = y^2 (12)$$

$$16 + 2y^4 = 12y^2$$

$$2y^4 - 12y^2 + 16 = 0$$

Substitute $u = y^2$

$$2u^2 - 12u + 16 = 0$$

$$2(u^2 - 6u + 8) = 0$$

$$2(u - 4)(u - 2) = 0$$

$$u = 4, 2$$

Re-substitute $u = y^2$

$$y^2 = 4 \quad y^2 = 2$$

$$y = \pm 2 \quad y = \pm \sqrt{2}$$

So since we have four different y values we will have to find x for each one. We

substitute these in to $x = \frac{4}{y}$ to get

$$x = \frac{4}{2}$$

$$= 2$$

$$x = \frac{4}{-2}$$

$$= -2$$

$$x = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{2}$$

$$= 2\sqrt{2}$$

$$x = \frac{4}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

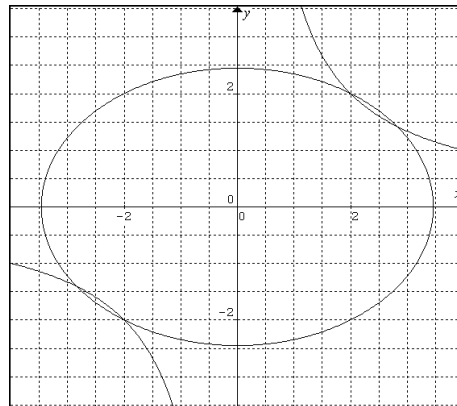
$$= -\frac{4\sqrt{2}}{2}$$

$$= -2\sqrt{2}$$

So we have four solutions, $(2, 2)$, $(-2, -2)$, $(2\sqrt{2}, \sqrt{2})$ and $(-2\sqrt{2}, -\sqrt{2})$.

Lets verify with a graph. We have an ellipse and a basic function ($xy = 4 \Rightarrow y = \frac{4}{x}$).

So we have



Example 2:

Solve the system.

a. $y = 3^x$
 $y = 3^{2x} - 2$

b. $y = \log_2(x+1)$
 $y = 5 - \log_2(x-3)$

Solution:

- a. This time solving is a little more complicated. There are a variety of direction we could go, however, we are going to start by noticing $3^{2x} = (3^x)^2$. So our system really is

$$y = 3^x$$

$$y = (3^x)^2 - 2$$

We can see clearly we will substitute the top equation into the bottom. That is, put y in for 3^x . We get

$$y = (y)^2 - 2$$

$$y = y^2 - 2$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

$$y = 2, -1$$

Now we substitute these values back in to get

$$2 = 3^x$$

$$-1 = 3^x$$

$$x = \log_3 2$$

However, the second equation is impossible (recall, exponential functions are always positive). Thus, that solution must be omitted. So we have a solution of $(\log_2 3, 2)$.

- b. Lastly, since these equations are both already solved for y, we can simply set them equal to one another. Then we are left with a logarithmic equation to solve. We get

$$\log_2(x+1) = 5 - \log_2(x-3)$$

$$\log_2(x+1) + \log_2(x-3) = 5$$

$$\log_2(x+1)(x-3) = 5$$

$$(x+1)(x-3) = 2^5$$

$$x^2 - 2x - 3 = 32$$

$$x^2 - 2x - 35 = 0$$

$$(x-7)(x+5) = 0$$

$$x = 7, -5$$

However, -5 cannot be a solution since it doesn't even check in the equation. Thus we only have $x = 7$. Now we substitute this back into either original equation to get the y value. We choose the first equation.

$$y = \log_2(7+1)$$

$$= \log_2 8$$

$$= 3$$

So the solution is $(7, 3)$.

13.4 Exercises

Solve the systems.

$$\begin{array}{l} 1. \quad x^2 + y^2 = 2 \\ \quad x + y = 2 \end{array}$$

$$\begin{array}{l} 2. \quad x^2 + y^2 = 25 \\ \quad y - x = 1 \end{array}$$

$$\begin{array}{l} 3. \quad 25x^2 + 9y^2 = 225 \\ \quad 5x + 3y = 15 \end{array}$$

$$\begin{array}{l} 4. \quad 9x^2 + 4y^2 = 36 \\ \quad 3x + 2y = 6 \end{array}$$

$$\begin{array}{l} 5. \quad y^2 = x + 3 \\ \quad 2y = x + 4 \end{array}$$

$$\begin{array}{l} 6. \quad y = x^2 \\ \quad 3x = y + 2 \end{array}$$

$$\begin{array}{l} 7. \quad x^2 - y^2 = 16 \\ \quad x - 2y = 1 \end{array}$$

$$\begin{array}{l} 8. \quad x^2 + 4y^2 = 25 \\ \quad x + 2y = 7 \end{array}$$

$$\begin{array}{l} 9. \quad x^2 + y^2 = 18 \\ \quad 2x + y = 3 \end{array}$$

$$\begin{array}{l} 10. \quad x^2 - y = 3 \\ \quad 2x - y = 3 \end{array}$$

$$\begin{array}{l} 11. \quad x^2 + y^2 = 20 \\ \quad y = x^2 \end{array}$$

$$\begin{array}{l} 12. \quad x^2 - y^2 = 3 \\ \quad y = x^2 - 3 \end{array}$$

$$\begin{array}{l} 13. \quad x^2 - x - y = 2 \\ \quad 4x - 3y = 0 \end{array}$$

$$\begin{array}{l} 14. \quad x^2 - 2x + 2y^2 = 8 \\ \quad 2x + y = 6 \end{array}$$

$$\begin{array}{l} 15. \quad x^2 + y^2 = 13 \\ \quad y = x^2 - 1 \end{array}$$

$$\begin{array}{l} 16. \quad x^2 - y = 5 \\ \quad x^2 + y^2 = 25 \end{array}$$

$$\begin{array}{l} 17. \quad x^2 + y^2 = 25 \\ \quad 2x^2 - 3y^2 = 5 \end{array}$$

$$\begin{array}{l} 18. \quad x^2 + y^2 = 4 \\ \quad 9x^2 + 16y^2 = 144 \end{array}$$

$$\begin{array}{l} 19. \quad x^2 + y^2 = 13 \\ \quad x^2 - y^2 = -16 \end{array}$$

$$\begin{array}{l} 20. \quad x^2 + y^2 = 16 \\ \quad y^2 - 2y^2 = 10 \end{array}$$

$$\begin{array}{l} 21. \quad x^2 + y^2 = 20 \\ \quad x^2 - y^2 = -12 \end{array}$$

$$\begin{array}{l} 22. \quad x^2 + y^2 = 14 \\ \quad x^2 - y^2 = 4 \end{array}$$

$$\begin{array}{l} 23. \quad xy = -\frac{9}{2} \\ \quad 3x + 2y = 6 \end{array}$$

$$\begin{array}{l} 24. \quad x + y = -6 \\ \quad xy = -7 \end{array}$$

$$\begin{array}{l} 25. \quad y = x^2 - 4 \\ \quad x^2 - y^2 = -16 \end{array}$$

$$\begin{array}{l} 26. \quad x^2 + y^2 = 25 \\ \quad y^2 = x + 5 \end{array}$$

$$\begin{array}{l} 27. \quad xy = \frac{1}{6} \\ \quad y + x = 5xy \end{array}$$

$$\begin{array}{l} 28. \quad xy = \frac{1}{12} \\ \quad x + y = 7xy \end{array}$$

$$\begin{array}{l} 29. \quad x^2 + xy + 2y^2 = 7 \\ \quad x - 2y = 5 \end{array}$$

$$\begin{array}{l} 30. \quad x^2 - xy + 3y^2 = 27 \\ \quad x - y = 2 \end{array}$$

$$\begin{array}{l} 31. \quad x^2 + y^2 = 5 \\ \quad xy = 2 \end{array}$$

$$\begin{array}{l} 32. \quad x^2 + y^2 = 20 \\ \quad xy = 8 \end{array}$$

$$\begin{array}{l} 33. \quad 3xy + x^2 = 34 \\ \quad 2xy - 3x^2 = 8 \end{array}$$

$$\begin{array}{l} 34. \quad 2xy + 3y^2 = 7 \\ \quad 3xy - 2y^2 = 4 \end{array}$$

$$\begin{array}{l} 35. \quad \frac{1}{x} + \frac{1}{y} = 5 \\ \quad \frac{1}{x} - \frac{1}{y} = -3 \end{array}$$

$$\begin{array}{l} 36. \quad \frac{1}{x} - \frac{1}{y} = 4 \\ \quad \frac{1}{x} + \frac{1}{y} = -2 \end{array}$$

$$37. \begin{aligned} \frac{2}{x^2} + \frac{5}{y^2} &= 3 \\ \frac{3}{x^2} - \frac{2}{y^2} &= 1 \end{aligned}$$

$$38. \begin{aligned} \frac{1}{x^2} - \frac{3}{y^2} &= 14 \\ \frac{2}{x^2} + \frac{1}{y^2} &= 35 \end{aligned}$$

$$39. \begin{aligned} x^4 &= y - 1 \\ y - 3x^2 + 1 &= 0 \end{aligned}$$

$$40. \begin{aligned} x^3 - y &= 0 \\ xy - 16 &= 0 \end{aligned}$$

$$41. \begin{aligned} y &= -\sqrt{x} \\ (x-3)^2 + y^2 &= 4 \end{aligned}$$

$$42. \begin{aligned} y &= \sqrt{x} \\ (x+2)^2 + y^2 &= 1 \end{aligned}$$

$$43. \begin{aligned} y &= 2^x \\ y &= 2^{2x} - 12 \end{aligned}$$

$$44. \begin{aligned} y &= 5^x \\ y &= 5^{2x} - 1 \end{aligned}$$

$$45. \begin{aligned} y &= e^{4x} \\ y &= e^{2x} + 6 \end{aligned}$$

$$46. \begin{aligned} y &= 2e^{2x} \\ y &= e^x - 1 \end{aligned}$$

$$47. \begin{aligned} y &= \log_2(x+4) \\ y &= 2 - \log_2(x+1) \end{aligned}$$

$$48. \begin{aligned} y &= \log_6 x \\ y &= -\log_6(x+1) \end{aligned}$$

$$49. \begin{aligned} y &= \log_9(x+1) \\ y &= \log_9 x + \frac{1}{2} \end{aligned}$$

$$50. \begin{aligned} y &= \log_{16}(x+3) \\ y &= \log_{16}(x-1) + \frac{1}{2} \end{aligned}$$