13.4 Nonlinear Systems

In this final section, we want to learn how to solve systems of equations that are not necessarily all linear. We call these non-linear systems of equations.

**Definition: Non-linear system of equations**

A system of equations where one or more equations involved is not a line.

We primarily use the substitution method to solve a non-linear system. However, sometimes elimination will work as well.

Also, just as before, the solution to a non-linear system is all the points of intersection of the graphs of the equations. Therefore, since we now have more that just lines, we can have a variety of numbers of solutions. The graph can intersect once, twice, several times or not at all. Therefore, we should always verify the solution(s) to a system by looking at the graph.

**Example 1:**

Solve the system.

a. \( x^2 + y^2 = 100 \)
   \( y - x = 2 \)

b. \( x^2 + y^2 = 25 \)
   \( y^2 = x + 5 \)

c. \( x^2 + 2y^2 = 12 \)
   \( xy = 4 \)

Solution:

a. We will solve this system by substitution. So we start by solving the bottom equation for \( y \) and then substitute it into the top equation. We get

\[
\begin{align*}
y - x &= 2 \implies y = x + 2 \\
x^2 + y^2 &= 100 \implies x^2 + (x + 2)^2 = 100 \\
x^2 + x^2 + 4x + 4 &= 100 \\
2x^2 + 4x - 96 &= 0 \\
2(x^2 + 2x - 48) &= 0 \\
2(x + 8)(x - 6) &= 0 \\
&= x = -8, 6
\end{align*}
\]

So we have two different \( x \) values. This means we should have two points of intersection. Lets now find the \( y \) values and verify with a graph.

\[
\begin{align*}
y &= x + 2 \\
y &= -8 + 2 \\
y &= -6 \\
y &= 6 + 2 \\
y &= 8
\end{align*}
\]

So our solutions are \(( -8, -6 )\) and \(( 6, 8 )\). We clearly have a circle and a line, thus we can easily graph them together and get
b. Again we will use substitution to solve. This time notice that the bottom equation is already solved for \( y^2 \) and we have a \( y^2 \) in the top equation. Thus, that is the substitution we will make. We get

\[
x^2 + y^2 = 25 \\
\Rightarrow x^2 + (x + 5) = 25
\]

Now we solve for \( x \) and then solve for \( y \).

\[
x^2 + (x + 5) = 25 \\
x^2 + x - 20 = 0 \\
(x + 5)(x - 4) = 0
\]

\[x = -5, 4\]

We substitute these back in to get \( y \). We get

\[
y^2 = -5 + 5 \\
y^2 = 4 + 5 \\
y^2 = 0 \\
y^2 = 9 \\
y = 0 \\
y = \pm 3
\]

So we have three different solutions \((-5, 0), (4, 3)\) and \((4, -3)\). Let's verify with a graph. We have here a parabola and a circle. We get

![Graph](image)

\[c. \text{ Again, we will use substitution to solve. We need to decide which variable to solve for first. It seems that } x \text{ or } y \text{ on the bottom equation would be easiest. So we will solve for } x. \text{ We get}
\]

\[xy = 4 \Rightarrow x = \frac{4}{y}\]

Now we substitute that into the first equation and solve. We have

\[
x^2 + 2y^2 = 12 \Rightarrow \left(\frac{4}{y}\right)^2 + 2y^2 = 12 \\
\]

\[
\frac{16}{y^2} + 2y^2 = 12
\]

We will have to clear the fractions and solve as we did in chapter 10, that is, using a substitution.
\[
\frac{16}{y^2} + 2y^2 = y^2(12)
\]
\[
16 + 2y^4 = 12y^2
\]
\[
2y^4 - 12y^2 + 16 = 0
\]
Substitute \(u = y^2\)
\[
2u^2 - 12u + 16 = 0
\]
\[
2(u^2 - 6u + 8) = 0
\]
\[
2(u - 4)(u - 2) = 0
\]
\[
u = 4, 2
\]
Re-substitute \(u = y^2\)
\[
y^2 = 4\]
\[
y^2 = 2
\]
\[
y = \pm 2\]
\[
y = \pm \sqrt{2}
\]
So since we have four different \(y\) values we will have to find \(x\) for each one. We substitute these in to \(x = \frac{4}{y}\) to get
\[
x = \frac{4}{2} = 2
\]
\[
x = \frac{4}{-2} = -2
\]
\[
x = \frac{4\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}
\]
\[
x = \frac{-4\sqrt{2}}{2} = -2\sqrt{2}
\]
So we have four solutions, \((2, 2), (-2, -2), (2\sqrt{2}, \sqrt{2})\) and \((-2\sqrt{2}, -\sqrt{2})\).

Let's verify with a graph. We have an ellipse and a basic function \((xy = 4 \Rightarrow y = \frac{4}{x})\).

So we have

Example 2:

Solve the system.

a. \(y = 3^x\) \quad b. \(y = \log_2(x + 1)\)

\[y = 3^{2x} - 2\] \quad \[y = 5 - \log_2(x - 3)\]
Solution:

a. This time solving is a little more complicated. There are a variety of direction we could go, however, we are going to start by noticing $3^{2x} = \left(3^x\right)^2$. So our system really is

$$
y = 3^x$$

$$
y = \left(3^x\right)^2 - 2$$

We can see clearly we will substitute the top equation into the bottom. That is, put $y$ in for $3^x$. We get

$$
y = (y)^2 - 2$$

$$
y = y^2 - 2$$

$$
y^2 - y - 2 = 0$$

$$
(y - 2)(y + 1) = 0$$

$$
y = 2, -1$$

Now we substitute these values back in to get

$$2 = 3^x$$

$$-1 = 3^x$$

However, the second equation is impossible (recall, exponential functions are always positive). Thus, that solution must be omitted. So we have a solution of $(\log_2 3, 2)$.

b. Lastly, since these equations are both already solved for $y$, we can simply set them equal to one another. Then we are left with a logarithmic equation to solve. We get

$$
\log_2(x + 1) = 5 - \log_2(x - 3)
$$

$$
\log_2(x + 1) + \log_2(x - 3) = 5
$$

$$
\log_2((x + 1)(x - 3)) = 5
$$

$$
(x + 1)(x - 3) = 2^5
$$

$$
x^2 - 2x - 3 = 32
$$

$$
x^2 - 2x - 35 = 0
$$

$$
(x - 7)(x + 5) = 0
$$

$$
x = 7, -5
$$

However, -5 cannot be a solution since it doesn't even check in the equation. Thus we only have $x = 7$. Now we substitute this back into either original equation to get the $y$ value. We choose the first equation.

$$
y = \log_2(7 + 1)
$$

$$
= \log_2 8
$$

$$
= 3
$$

So the solution is $(7, 3)$.

13.4 Exercises

Solve the systems.
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<td>1.</td>
<td>( x^2 + y^2 = 2 )</td>
<td>( x + y = 2 )</td>
</tr>
<tr>
<td>2.</td>
<td>( x^2 + y^2 = 25 )</td>
<td>( y - x = 1 )</td>
</tr>
<tr>
<td>3.</td>
<td>( 25x^2 + 9y^2 = 225 )</td>
<td>( 5x + 3y = 15 )</td>
</tr>
<tr>
<td>4.</td>
<td>( 9x^2 + 4y^2 = 36 )</td>
<td>( 3x + 2y = 6 )</td>
</tr>
<tr>
<td>5.</td>
<td>( y^2 = x + 3 )</td>
<td>( 2y = x + 4 )</td>
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<tr>
<td>6.</td>
<td>( y = x^2 )</td>
<td>( 3x = y + 2 )</td>
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<tr>
<td>7.</td>
<td>( x^2 - y^2 = 16 )</td>
<td>( x - 2y = 1 )</td>
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<tr>
<td>8.</td>
<td>( x^2 + 4y^2 = 25 )</td>
<td>( x + 2y = 7 )</td>
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<tr>
<td>9.</td>
<td>( x^2 + y^2 = 18 )</td>
<td>( 2x + y = 3 )</td>
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<td>( x^2 - y = 3 )</td>
<td>( 2x - y = 3 )</td>
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<td>11.</td>
<td>( x^2 + y^2 = 20 )</td>
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<td>12.</td>
<td>( x^2 - y^2 = 3 )</td>
<td>( y = x^2 - 3 )</td>
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<tr>
<td>13.</td>
<td>( x^2 - x - y = 2 )</td>
<td>( 4x - 3y = 0 )</td>
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<td>14.</td>
<td>( x^2 - 2x + 2y^2 = 8 )</td>
<td>( 2x + y = 6 )</td>
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<tr>
<td>15.</td>
<td>( x^2 + y^2 = 13 )</td>
<td>( y = x^2 - 1 )</td>
</tr>
<tr>
<td>16.</td>
<td>( x^2 - y = 5 )</td>
<td>( x^2 + y^2 = 25 )</td>
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<tr>
<td>17.</td>
<td>( x^2 + y^2 = 25 )</td>
<td>( 2x^2 - 3y^2 = 5 )</td>
</tr>
<tr>
<td>18.</td>
<td>( x^2 + y^2 = 4 )</td>
<td>( 9x^2 + 16y^2 = 144 )</td>
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<tr>
<td>19.</td>
<td>( x^2 + y^2 = 13 )</td>
<td>( x^2 - y^2 = -16 )</td>
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<tr>
<td>20.</td>
<td>( x^2 + y^2 = 16 )</td>
<td>( y^2 - 2y^2 = 10 )</td>
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<tr>
<td>21.</td>
<td>( x^2 + y^2 = 20 )</td>
<td>( x^2 - y^2 = -12 )</td>
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<tr>
<td>22.</td>
<td>( x^2 + y^2 = 14 )</td>
<td>( x^2 - y^2 = 4 )</td>
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<tr>
<td>23.</td>
<td>( xy = -\frac{9}{2} )</td>
<td>( 3x + 2y = 6 )</td>
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<tr>
<td>24.</td>
<td>( x + y = -6 )</td>
<td>( xy = -7 )</td>
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<tr>
<td>25.</td>
<td>( y = x^2 - 4 )</td>
<td>( x^2 - y^2 = -16 )</td>
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<tr>
<td>26.</td>
<td>( x^2 + y^2 = 25 )</td>
<td>( y^2 = x + 5 )</td>
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<tr>
<td>27.</td>
<td>( xy = \frac{1}{2} )</td>
<td>( y + x = 5xy )</td>
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<tr>
<td>28.</td>
<td>( xy = \frac{1}{12} )</td>
<td>( x + y = 7xy )</td>
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<tr>
<td>29.</td>
<td>( x^2 + xy + 2y^2 = 7 )</td>
<td>( x - 2y = 5 )</td>
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<tr>
<td>30.</td>
<td>( x^2 - xy + 3y^2 = 27 )</td>
<td>( x - y = 2 )</td>
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<tr>
<td>31.</td>
<td>( x^2 + y^2 = 5 )</td>
<td>( xy = 2 )</td>
</tr>
<tr>
<td>32.</td>
<td>( x^2 + y^2 = 20 )</td>
<td>( xy = 8 )</td>
</tr>
<tr>
<td>33.</td>
<td>( 3xy + x^2 = 34 )</td>
<td>( 2xy - 3x^2 = 8 )</td>
</tr>
<tr>
<td>34.</td>
<td>( 2xy + 3y^2 = 7 )</td>
<td>( 3xy - 2y^2 = 4 )</td>
</tr>
<tr>
<td>35.</td>
<td>( \frac{1}{x} + \frac{1}{y} = 5 )</td>
<td>( \frac{1}{x} - \frac{1}{y} = -3 )</td>
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<tr>
<td>36.</td>
<td>( \frac{1}{x} - \frac{1}{y} = 4 )</td>
<td>( \frac{1}{x} + \frac{1}{y} = -2 )</td>
</tr>
</tbody>
</table>
37. \( \frac{2}{x^2} + \frac{5}{y^2} = 3 \)
\( \frac{3}{x^2} - \frac{2}{y^2} = 1 \)

40. \( x^3 - y = 0 \)
\( xy - 16 = 0 \)

43. \( y = 2^x \)
\( y = 2^{2x} - 12 \)

46. \( y = 2e^{2x} \)
\( y = e^r - 1 \)

49. \( y = \log_9(x + 1) \)
\( y = \log_9(x + \frac{1}{2}) \)

38. \( \frac{1}{x^2} - \frac{3}{y^2} = 14 \)
\( \frac{2}{x^2} + \frac{1}{y^2} = 35 \)

41. \( y = -\sqrt{x} \)
\( (x - 3)^2 + y^2 = 4 \)

44. \( y = 5^x \)
\( y = 5^{2x} - 1 \)

47. \( y = \log_2(x + 4) \)
\( y = 2 - \log_2(x + 1) \)

49. \( y = \log_9(x + 1) \)
\( y = \log_9(x + \frac{1}{2}) \)

39. \( x^4 = y - 1 \)
\( y - 3x^2 + 1 = 0 \)

42. \( y = \sqrt{x} \)
\( (x + 2)^2 + y^2 = 1 \)

45. \( y = e^{4x} \)
\( y = e^{2x} + 6 \)

48. \( y = \log_6 x \)
\( y = -\log_6(x + 1) \)