13.4 Nonlinear Systems

In this final section, we want to learn how to solve systems of equations that are not necessarily all linear. We call these <u>non-linear systems of equations</u>.

Definition: Non-linear system of equations
A system of equations where one or more equations involved is not a line.

We primarily use the substitution method to solve a non-linear system. However, sometimes elimination will work as well.

Also, just as before, the solution to a non-linear system is all the points of intersection of the graphs of the equations. Therefore, since we now have more that just lines, we can have a variety of numbers of solutions. The graph can intersect once, twice, several times or not at all. Therefore, we should always verify the solution(s) to a system by looking at the graph.

<u>Example 1</u>:

Solve the system.

a. $\begin{array}{l} x^2+y^2=100 \\ y-x=2 \end{array}$ b. $\begin{array}{l} x^2+y^2=25 \\ y^2=x+5 \end{array}$ c. $\begin{array}{l} x^2+2y^2=12 \\ xy=4 \end{array}$

Solution:

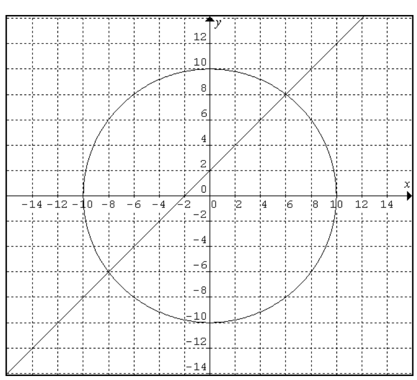
a. We will solve this system by substitution. So we start by solving the bottom equation for y and then substitute it into the top equation. We get

$$y-x=2 \Rightarrow y=x+2$$
$$x^2+y^2=100 \Rightarrow x^2+(x+2)^2=100$$
$$x^2+x^2+4x+4=100$$
$$2x^2+4x-96=0$$
$$2(x^2+2x-48)=0$$
$$2(x+8)(x-6)=0$$
$$x=-8,\ 6$$

So we have two different x values. This means we should have two points of intersection. Lets now find the y values and verify with a graph.

$$\begin{array}{ll} y=x+2 & y=x+2 \\ y=-8+2 & y=6+2 \\ y=-6 & y=8 \end{array}$$

So our solutions are $(-8,-6)$ and $(6,8)$. We clearly have a circle and a line, thus we can easily graph them together and get



b. Again we will use substitution to solve. This time notice that the bottom equation is already solved for y^2 and we have a y^2 in the top equation. Thus, that is the substitution we will make. We get

$$\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= \overbrace{x+5} \end{aligned} \quad \Rightarrow x^2 + (x+5) = 25$$

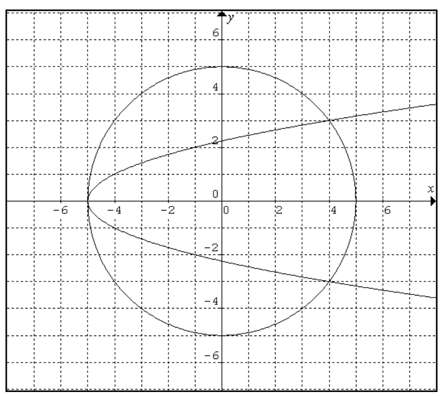
Now we solve for x and then solve for y.

$$\begin{aligned} x^2 + (x+5) &= 25 \\ x^2 + x - 20 &= 0 \\ (x+5)(x-4) &= 0 \\ x &= -5,\ 4 \end{aligned}$$

We substitute these back in to get y. We get

$$\begin{aligned} y^2 &= -5+5 \\ y^2 &= 0 \\ y &= 0 \end{aligned} \qquad\qquad \begin{aligned} y^2 &= 4+5 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

So we have three different solutions $(-5,\ 0)$, $(4,\ 3)$ and $(4,\ -3)$. Lets verify with a graph. We have here a parabola and a circle. We get



c. Again, we will use substitution to solve. We need to decide which variable to solve for first. It seems that x or y on the bottom equation would be easiest. So we will solve for x. We get

$$xy = 4 \Rightarrow x = \frac{4}{y}$$

Now we substitute that into the first equation and solve. We have

$$\begin{aligned} x^2 + 2y^2 = 12 \Rightarrow \left(\frac{4}{y}\right)^2 + 2y^2 &= 12 \\ \frac{16}{y^2} + 2y^2 &= 12 \end{aligned}$$

We will have to clear the fractions and solve as we did in chapter 10, that is, using a substitution.

$$y^2\left(\frac{16}{y^2}+2y^2\right)=y^2(12)$$
$$16+2y^4=12y^2$$
$$2y^4-12y^2+16=0$$
Substitute u = y^2
$$2u^2-12u+16=0$$
$$2(u^2-6u+8)=0$$
$$2(u-4)(u-2)=0$$
$$u=4,\ 2$$
Re-substitute u = y^2
$$y^2=4 \qquad y^2=2$$
$$y=\pm 2 \qquad y=\pm\sqrt{2}$$

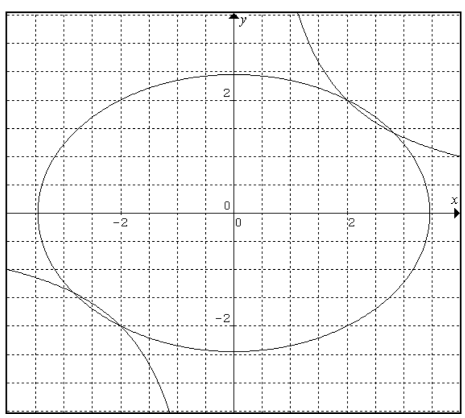
So since we have four different y values we will have to find x for each one. We substitute these in to $x=\frac{4}{y}$ to get

$$x=\frac{4}{2}=2 \qquad x=\frac{4}{-2}=-2 \qquad x=\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{4\sqrt{2}}{2}=2\sqrt{2} \qquad x=\frac{4}{-\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\frac{4\sqrt{2}}{2}=-2\sqrt{2}$$

So we have four solutions, $(2,\ 2)$, $(-2,\ -2)$, $(2\sqrt{2},\ \sqrt{2})$ and $(-2\sqrt{2},\ -\sqrt{2})$.

Lets verify with a graph. We have an ellipse and a basic function ($xy=4 \Rightarrow y=\frac{4}{x}$).

So we have



Example 2:

Solve the system.

a. $y=3^x$
$y=3^{2x}-2$

b. $y=\log_2(x+1)$
$y=5-\log_2(x-3)$

Solution:

a. This time solving is a little more complicated. There are a variety of direction we could go, however, we are going to start by noticing $3^{2x} = (3^x)^2$. So our system really is

$$y = 3^x$$
$$y = (3^x)^2 - 2$$

We can see clearly we will substitute the top equation into the bottom. That is, put y in for 3^x. We get

$$\begin{aligned} y &= (y)^2 - 2 \\ y &= y^2 - 2 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \\ y &= 2, -1 \end{aligned}$$

Now we substitute these values back in to get

$$\begin{aligned} 2 &= 3^x \\ x &= \log_3 2 \end{aligned} \qquad -1 = 3^x$$

However, the second equation is impossible (recall, exponential functions are always positive). Thus, that solution must be omitted. So we have a solution of $(\log_2 3, 2)$.

b. Lastly, since these equations are both already solved for y, we can simply set them equal to one another. Then we are left with a logarithmic equation to solve. We get

$$\begin{aligned} \log_2(x+1) &= 5 - \log_2(x-3) \\ \log_2(x+1) + \log_2(x-3) &= 5 \\ \log_2(x+1)(x-3) &= 5 \\ (x+1)(x-3) &= 2^5 \\ x^2 - 2x - 3 &= 32 \\ x^2 - 2x - 35 &= 0 \\ (x-7)(x+5) &= 0 \\ x &= 7, -5 \end{aligned}$$

However, -5 cannot be a solution since is doesn't even check in the equation. Thus we only have x = 7. Now we substitute this back into either original equation to get the y value. We choose the first equation.

$$\begin{aligned} y &= \log_2(7+1) \\ &= \log_2 8 \\ &= 3 \end{aligned}$$

So the solution is $(7, 3)$.

13.4 Exercises

Solve the systems.

1. $\begin{aligned} x^2+y^2&=2 \\ x+y&=2 \end{aligned}$

2. $\begin{aligned} x^2+y^2&=25 \\ y-x&=1 \end{aligned}$

3. $\begin{aligned} 25x^2+9y^2&=225 \\ 5x+3y&=15 \end{aligned}$

4. $\begin{aligned} 9x^2+4y^2&=36 \\ 3x+2y&=6 \end{aligned}$

5. $\begin{aligned} y^2&=x+3 \\ 2y&=x+4 \end{aligned}$

6. $\begin{aligned} y&=x^2 \\ 3x&=y+2 \end{aligned}$

7. $\begin{aligned} x^2-y^2&=16 \\ x-2y&=1 \end{aligned}$

8. $\begin{aligned} x^2+4y^2&=25 \\ x+2y&=7 \end{aligned}$

9. $\begin{aligned} x^2+y^2&=18 \\ 2x+y&=3 \end{aligned}$

10. $\begin{aligned} x^2-y&=3 \\ 2x-y&=3 \end{aligned}$

11. $\begin{aligned} x^2+y^2&=20 \\ y&=x^2 \end{aligned}$

12. $\begin{aligned} x^2-y^2&=3 \\ y&=x^2-3 \end{aligned}$

13. $\begin{aligned} x^2-x-y&=2 \\ 4x-3y&=0 \end{aligned}$

14. $\begin{aligned} x^2-2x+2y^2&=8 \\ 2x+y&=6 \end{aligned}$

15. $\begin{aligned} x^2+y^2&=13 \\ y&=x^2-1 \end{aligned}$

16. $\begin{aligned} x^2-y&=5 \\ x^2+y^2&=25 \end{aligned}$

17. $\begin{aligned} x^2+y^2&=25 \\ 2x^2-3y^2&=5 \end{aligned}$

18. $\begin{aligned} x^2+y^2&=4 \\ 9x^2+16y^2&=144 \end{aligned}$

19. $\begin{aligned} x^2+y^2&=13 \\ x^2-y^2&=-16 \end{aligned}$

20. $\begin{aligned} x^2+y^2&=16 \\ y^2-2y^2&=10 \end{aligned}$

21. $\begin{aligned} x^2+y^2&=20 \\ x^2-y^2&=-12 \end{aligned}$

22. $\begin{aligned} x^2+y^2&=14 \\ x^2-y^2&=4 \end{aligned}$

23. $\begin{aligned} xy&=-\tfrac{9}{2} \\ 3x+2y&=6 \end{aligned}$

24. $\begin{aligned} x+y&=-6 \\ xy&=-7 \end{aligned}$

25. $\begin{aligned} y&=x^2-4 \\ x^2-y^2&=-16 \end{aligned}$

26. $\begin{aligned} x^2+y^2&=25 \\ y^2&=x+5 \end{aligned}$

27. $\begin{aligned} xy&=\tfrac{1}{6} \\ y+x&=5xy \end{aligned}$

28. $\begin{aligned} xy&=\tfrac{1}{12} \\ x+y&=7xy \end{aligned}$

29. $\begin{aligned} x^2+xy+2y^2&=7 \\ x-2y&=5 \end{aligned}$

30. $\begin{aligned} x^2-xy+3y^2&=27 \\ x-y&=2 \end{aligned}$

31. $\begin{aligned} x^2+y^2&=5 \\ xy&=2 \end{aligned}$

32. $\begin{aligned} x^2+y^2&=20 \\ xy&=8 \end{aligned}$

33. $\begin{aligned} 3xy+x^2&=34 \\ 2xy-3x^2&=8 \end{aligned}$

34. $\begin{aligned} 2xy+3y^2&=7 \\ 3xy-2y^2&=4 \end{aligned}$

35. $\begin{aligned} \frac{1}{x}+\frac{1}{y}&=5 \\ \frac{1}{x}-\frac{1}{y}&=-3 \end{aligned}$

36. $\begin{aligned} \frac{1}{x}-\frac{1}{y}&=4 \\ \frac{1}{x}+\frac{1}{y}&=-2 \end{aligned}$

37. $\frac{2}{x^2}+\frac{5}{y^2}=3$, $\frac{3}{x^2}-\frac{2}{y^2}=1$

38. $\frac{1}{x^2}-\frac{3}{y^2}=14$, $\frac{2}{x^2}+\frac{1}{y^2}=35$

39. $x^4=y-1$, $y-3x^2+1=0$

40. $x^3-y=0$, $xy-16=0$

41. $y=-\sqrt{x}$, $(x-3)^2+y^2=4$

42. $y=\sqrt{x}$, $(x+2)^2+y^2=1$

43. $y=2^x$, $y=2^{2x}-12$

44. $y=5^x$, $y=5^{2x}-1$

45. $y=e^{4x}$, $y=e^{2x}+6$

46. $y=2e^{2x}$, $y=e^x-1$

47. $y=\log_2(x+4)$, $y=2-\log_2(x+1)$

48. $y=\log_6 x$, $y=-\log_6(x+1)$

49. $y=\log_9(x+1)$, $y=\log_9 x+\frac{1}{2}$

50. $y=\log_{16}(x+3)$, $y=\log_{16}(x-1)+\frac{1}{2}$