

13.3 Using Cramer's Rule to Solve Systems

Now that we can solve 2x2 and 3x3 systems of equations, we want to learn another technique for solving these systems. This new technique will require us to get familiar with several new concepts. Let's start with the following definition.

Definitions: Matrix
A matrix is a rectangular array of numbers written inside of brackets. Each number in a matrix is called an entry , each horizontal set of numbers is called a row and each vertical set of numbers is called a column .

Matrices come in a wide variety of sizes. When writing the size of a matrix, we always list the rows first. So a 2x3 matrix would have 2 rows and 3 columns, for example.

Let's get a handle on these ideas with the following example.

Example 1:

List the entries of the following matrix. What is the 3rd entry of the 2nd row and the 1st entry of the 3rd column?

$$\begin{bmatrix} 1 & 5 & -2 & 6 & -8 \\ 0 & -3 & 2 & 7 & -6 \\ 10 & 0 & -4 & -1 & 3 \end{bmatrix}$$

Solution:

Since this matrix has 3 rows and 5 columns, it is a 3x5 matrix.

So, with this in mind, the 3rd entry of the 2nd row would be a 2, and the 1st entry of the 3rd column would be -2.

With the basic idea of a matrix now down, we need to talk about a couple of different kinds of matrices. Namely, the coefficient matrix and the augmented matrix.

Definition: Coefficient Matrix
A matrix which is composed of all of the coefficients of a system of equations but does not include the constant terms.

Definition: Augmented Matrix
A matrix which is composed of all coefficients and constants of a system of equations.

These are fairly easy to find as we see in the next example.

Example 2:

Write the coefficient matrix and the augmented matrix of the following systems.

a.	$3x + 2y = 3$		$2x + y - z = 4$
	$9x - 8y = 0$	b.	$y + 3z = 2$
			$3x + 2y = 4$

Solution:

a. The coefficient matrix is the matrix that is generated by coefficients, but not the constant terms of the given system. The augmented matrix is generated by the coefficients and the constants of the system. So this means we must have

$$\begin{array}{l} 3x + 2y = 3 \\ 9x - 8y = 0 \end{array} \Rightarrow \begin{array}{cc} \left[\begin{array}{cc} 3 & 2 \\ 9 & -8 \end{array} \right] & \Rightarrow \left[\begin{array}{cc|c} 3 & 2 & 3 \\ 9 & -8 & 0 \end{array} \right] \\ \text{Coefficient Matrix} & \text{Augmented Matrix} \end{array}$$

b. Like in part a. above, we generate the coefficient matrix by only the coefficients and the augmented matrix by using the entire system. This gives us

$$\begin{array}{l} 2x + y - z = 4 \\ y + 3z = 2 \\ 3x + 2y = 4 \end{array} \Rightarrow \begin{array}{ccc} \left[\begin{array}{ccc} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & 2 & 0 \end{array} \right] & \Rightarrow & \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 3 & 2 & 0 & 4 \end{array} \right] \\ \text{Coefficient Matrix} & & \text{Augmented Matrix} \end{array}$$

There are a number of operations and other things we can do with a matrix, but for the sake of simplicity, let's take a look at only the operation that we need for solving systems. That is, the operation of the determinant.

Definition: Determinant

A number associated with a square matrix. This number has numerous uses in a variety of applications.

More important than the definition of a determinant is how we find a determinant. It varies slightly depending on the size of your matrix. We will only look at evaluating a 2x2 and a 3x3 determinant since that is the size of the systems we will be solving.

Evaluating a Determinant

Determinant of a 2x2 matrix:

The determinant of the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of a 3x3 matrix:

The determinant of the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Although there are a number of ways to calculate a 3x3 determinant, for the sake of simplicity, we are only giving one general formula. The reader may reference a precalculus, or linear algebra book for further development.

Evaluating a determinant boils down to remembering the formulas given above. Let's see how this works in the next example.

Example 3:

Find the determinant of each matrix.

a. $\begin{bmatrix} 3 & 2 \\ 9 & -8 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 4 & -2 & 1 \\ 3 & -2 & 0 \\ -5 & -2 & -3 \end{bmatrix}$

Solution:

a. To find the determinant of a matrix, we simply need to follow the formula given above.

$$\begin{aligned} \det\left(\begin{bmatrix} 3 & 2 \\ 9 & -8 \end{bmatrix}\right) &= \begin{vmatrix} 3 & 2 \\ 9 & -8 \end{vmatrix} \\ &= 3 \cdot (-8) - 9 \cdot 2 \\ &= -24 - 18 \\ &= -42 \end{aligned}$$

Notice that an easy way to remember the formula for a 2x2 determinant is to multiply along the diagonals and subtract those products. Just make sure you put the first entry in the matrix as the first product in the determinant.

b. Again, to find the determinant we just use the formula, that is, find the difference of the product of the diagonals. We get

$$\begin{aligned} \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} &= 1 \cdot 0 - 1 \cdot (-2) \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

c. To find the determinant of a 3x3 matrix, we also just apply the formula. It's just that the formula is much larger. However, there is a pattern to it. Notice the first row is the number in front of each "minor" determinant (with the pattern of + - + signs) and each "minor" determinant is what we get by simply crossing out the corresponding row and column of the first row we are using.

So, for our given matrix we get

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}\right) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 3 & 2 & 0 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}$$

Now we simply have to evaluate the smaller determinants and finish our computations.

$$\begin{aligned}
& 2 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} \\
&= 2(1 \cdot 0 - 2 \cdot 3) - 1(0 \cdot 0 - 3 \cdot 3) - 1(0 \cdot 2 - 3 \cdot 1) \\
&= 2(-6) - 1(-9) - 1(-3) \\
&= -12 + 9 + 3 \\
&= 0
\end{aligned}$$

d. Again, we will follow the formula

$$\begin{aligned}
\begin{vmatrix} 4 & -2 & 1 \\ 3 & -2 & 0 \\ -5 & -2 & -3 \end{vmatrix} &= 4 \cdot \begin{vmatrix} -2 & 0 \\ -2 & -3 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & 0 \\ -5 & -3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & -2 \\ -5 & -2 \end{vmatrix} \\
&= 4((-2)(-3) - (-2) \cdot 0) + 2(3 \cdot (-3) - (-5) \cdot 0) + 1(3 \cdot (-2) - (-5)(-2)) \\
&= 4(6 - 0) + 2(-9 - 0) + 1(-6 - 10) \\
&= 24 - 18 - 16 \\
&= -10
\end{aligned}$$

The determinant is a very powerful tool in matrices and can do numerous things. However, we are only interested in using the determinant to solve systems of equations.

To do this we use something called Cramer's Rule. This rule is named after 16th century Swiss mathematician Gabriel Cramer.

Cramer's Rule for Solving 2x2 Systems

Consider the system

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Let the three determinants D , D_x and D_y be defined as

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Then, if $D \neq 0$, the system has a unique solution of

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D}$$

Although Cramer's rule seems complicated, it's merely a matter of computing the coefficient matrix determinant and then computing that same determinant where each column is replaced by the constants in the system. Then, generating the fractions to get the solution.

Let's look at it with the next example.

Example 4:

Solve the system by using Cramer's Rule.

$$\begin{array}{l} \text{a.} \quad 5x - 9y = 7 \\ \quad -3x + 7y = -5 \end{array}$$

$$\begin{array}{l} \text{b.} \quad 2x + 3y = 0 \\ \quad x + y = -1 \end{array}$$

Solution:

a. The first this we need to do is determine all of the determinants D , D_x and D_y . We use the formulas given above to do so as follows.

$$\begin{array}{lll} D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} & D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ D = \begin{vmatrix} 5 & -9 \\ -3 & 7 \end{vmatrix} & D_x = \begin{vmatrix} 7 & -9 \\ -5 & 7 \end{vmatrix} & D_y = \begin{vmatrix} 5 & 7 \\ -3 & -5 \end{vmatrix} \end{array}$$

This gives

$$\begin{array}{lll} D = 5 \cdot 7 - (-3)(-9) & D_x = 7 \cdot 7 - (-5)(-9) & D_y = 5(-5) - (-3) \cdot 7 \\ = 35 - 27 & = 49 - 45 & = -25 + 21 \\ = 8 & = 4 & = -4 \end{array}$$

So then we just need to evaluate $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ to get our solution. These give

$$x = \frac{D_x}{D} = \frac{4}{8} = \frac{1}{2} \text{ and } y = \frac{D_y}{D} = \frac{-4}{8} = -\frac{1}{2}$$

So the solution is $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

b. As in part a. above, we need to determine the determinants in order to solve the system. This gives

$$\begin{array}{lll} D = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & D_x = \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} & D_y = \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} \\ = 2 - 3 & = 0 + 3 & = -2 - 0 \\ = -1 & = 3 & = -2 \end{array}$$

Calculating x and y we get $x = \frac{D_x}{D} = \frac{3}{-1} = -3$ and $y = \frac{D_y}{D} = \frac{-2}{-1} = 2$

So the solution is $(-3, 2)$.

Although solving a 2x2 system with Cramer's Rule is not too difficult, it is a bit more time consuming and labor intensive to do 3x3 systems as we see next.

Cramer's Rule for Solving 3x3 Systems

Consider the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let the four determinants D , D_x , D_y and D_z be defined as

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then, if $D \neq 0$, the system has a unique solution of

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad z = \frac{D_z}{D}$$

As above with the 2x2 case, this seems complicated. However, it's just a matter of calculating several determinants and plugging the values in.

Example 5:

Solve the system by using Cramer's Rule

$$x - y + 2z = -4$$

$$x + y + z = -2$$

a. $3x + y - 4z = -6$

b. $x - y + z = 0$

$$2x + 3y - 4z = 4$$

$$4x + 2y + z = 3$$

Solution:

a. The first thing we need to do here is to evaluate all of the corresponding determinants. We start with the coefficient determinant.

$$\begin{aligned} D &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & -4 \\ 2 & 3 & -4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -4 \\ 3 & -4 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -4 \\ 2 & -4 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ &= (-4 + 12) + (-12 + 8) + 2(9 - 2) \\ &= 8 - 4 + 14 \\ &= 18 \end{aligned}$$

Now we compute the other determinants

$$\begin{aligned}
 D_x &= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} -4 & -1 & 2 \\ -6 & 1 & -4 \\ 4 & 3 & -4 \end{vmatrix} = -4 \begin{vmatrix} 1 & -4 \\ 3 & -4 \end{vmatrix} - (-1) \begin{vmatrix} -6 & -4 \\ 4 & -4 \end{vmatrix} + 2 \begin{vmatrix} -6 & 1 \\ 4 & 3 \end{vmatrix} \\
 &= -4(-4 + 12) + (24 + 16) + 2(-18 - 4) \\
 &= -32 + 40 - 44 \\
 &= -36
 \end{aligned}$$

$$\begin{aligned}
 D_y &= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 3 & -6 & -4 \\ 2 & 4 & -4 \end{vmatrix} = 1 \begin{vmatrix} -6 & -4 \\ 4 & -4 \end{vmatrix} - (-4) \begin{vmatrix} 3 & -4 \\ 2 & -4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -6 \\ 2 & 4 \end{vmatrix} \\
 &= (24 + 16) + 4(-12 + 8) + 2(12 + 12) \\
 &= 40 - 16 + 48 \\
 &= 72
 \end{aligned}$$

$$\begin{aligned}
 D_z &= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -4 \\ 3 & 1 & -6 \\ 2 & 3 & 4 \end{vmatrix} = 1 \begin{vmatrix} -6 & -4 \\ 3 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -6 \\ 2 & 4 \end{vmatrix} + (-4) \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\
 &= (4 + 18) + (12 + 12) - 4(9 - 2) \\
 &= 22 + 24 - 28 \\
 &= 18
 \end{aligned}$$

Now that we have that, we simply need to determine the solutions by the formulas

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad z = \frac{D_z}{D}$$

We get

$$\begin{aligned}
 x &= \frac{-36}{18} & y &= \frac{72}{18} & z &= \frac{18}{18} \\
 &= -2 & &= 4 & &= 1
 \end{aligned}$$

So the solution to the system is $(-2, 4, 1)$.

b. As we did in part a. above, we need to start by determining all of the determinants required for Cramer's Rule. We get

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \quad D_x = \begin{vmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \quad D_y = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} \quad D_z = \begin{vmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 4 & 2 & 3 \end{vmatrix}$$

We will leave it to the reader to work through the details of these determinants. It can be shown that these determinants give us

$$D = 6 \qquad D_x = 12 \qquad D_y = -6 \qquad D_z = -18$$

Plugging in gives us

$$\begin{aligned} x &= \frac{D_x}{D} & y &= \frac{D_y}{D} & z &= \frac{D_z}{D} \\ &= \frac{12}{6} & &= \frac{-6}{6} & &= \frac{-18}{6} \\ &= 2 & &= -1 & &= -3 \end{aligned}$$

So the solution to the system is (2, -1, -3).

The only real “hang-up” with Cramer’s Rule is the unfortunate situation in which the denominator determinant ends up as zero. If this is the case, Cramer’s Rule does not work. You would have to go back and solve the system using one of the other methods that you have learned for solving a system.

13.3 Exercises

Write the coefficient matrix and the augmented matrix of the following systems.

$$\begin{array}{lll} 1. \quad \begin{array}{l} x + y = 4 \\ x - y = 2 \end{array} & 2. \quad \begin{array}{l} x - 2y = 0 \\ 2x - y = 6 \end{array} & 3. \quad \begin{array}{l} x - 2y = 5 \\ -3x + 6y = 4 \end{array} \end{array}$$

$$\begin{array}{lll} 4. \quad \begin{array}{l} x + y + 2z = 9 \\ y - \frac{7}{2}z = -\frac{17}{2} \\ z = 3 \end{array} & 5. \quad \begin{array}{l} x = 10 \\ 3x + 2y = 2 \\ x + y + 2z = 0 \end{array} & 6. \quad \begin{array}{l} x + z = 4 \\ y = 2 \\ 4x + z = 7 \end{array} \end{array}$$

Find the determinant of each matrix.

$$\begin{array}{lll} 7. \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} & 8. \quad \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} & 9. \quad \begin{bmatrix} -1 & 6 \\ -2 & -2 \end{bmatrix} \end{array}$$

$$\begin{array}{lll} 10. \quad \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix} & 11. \quad \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} & 12. \quad \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{lll} 13. \quad \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} & 14. \quad \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & 2 \\ 0 & -2 & 4 \end{bmatrix} & 15. \quad \begin{bmatrix} 0 & 1 & 0 \\ 5 & -1 & -2 \\ -3 & 4 & 0 \end{bmatrix} \end{array}$$

$$16. \begin{bmatrix} -5 & -1 & -3 \\ 6 & -3 & 2 \\ -5 & 4 & 1 \end{bmatrix}$$

$$17. \begin{bmatrix} -2 & -1 & 3 \\ 2 & -4 & 1 \\ 4 & 2 & -6 \end{bmatrix}$$

$$18. \begin{bmatrix} 1 & 10 & -10 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve the system by using Cramer's Rule.

$$19. \begin{aligned} x + y &= -1 \\ -x + y &= 1 \end{aligned}$$

$$20. \begin{aligned} 3x - y &= 6 \\ x + 3y &= 2 \end{aligned}$$

$$21. \begin{aligned} -2x + y &= 11 \\ -5x + y &= -19 \end{aligned}$$

$$22. \begin{aligned} -x + y &= -5 \\ 2x + y &= 4 \end{aligned}$$

$$23. \begin{aligned} x + y &= 1 \\ 3x - y &= -5 \end{aligned}$$

$$24. \begin{aligned} \frac{1}{2}x + y &= 0 \\ -x + y &= 3 \end{aligned}$$

$$25. \begin{aligned} 3x + 2y &= 6 \\ 2y &= -6 \end{aligned}$$

$$26. \begin{aligned} y &= 3x \\ 4y - 12x &= 8 \end{aligned}$$

$$27. \begin{aligned} y &= -6x \\ y &= x - 5 \end{aligned}$$

$$28. \begin{aligned} y &= x - 3 \\ 3x + 2y &= 9 \end{aligned}$$

$$29. \begin{aligned} 10x - 5y &= 7 \\ 2x - y &= 4 \end{aligned}$$

$$30. \begin{aligned} 2x - 5y &= 1 \\ 4x + 3y &= 0 \end{aligned}$$

$$31. \begin{aligned} x &= 3 \\ -x + y &= 3 \\ y + 2z &= 4 \end{aligned}$$

$$32. \begin{aligned} 2y + z &= 7 \\ 2x - z &= 3 \\ x - y &= 3 \end{aligned}$$

$$33. \begin{aligned} 3x + y &= 5 \\ 3y - z &= 2 \\ x + z &= 5 \end{aligned}$$

$$34. \begin{aligned} 2x + 3z &= 5 \\ 3y + 2z &= 3 \\ 3x + 4y &= -10 \end{aligned}$$

$$35. \begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x - z &= 0 \end{aligned}$$

$$36. \begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

$$37. \begin{aligned} x + y + 8z &= 3 \\ 2x + y + 11z &= 4 \\ x + 3z &= 0 \end{aligned}$$

$$38. \begin{aligned} 2x + y - z &= 4 \\ y + 3z &= 2 \\ 3x + 2y &= 4 \end{aligned}$$

$$39. \begin{aligned} 3x + y + z &= 2 \\ 4x + 2z &= 1 \\ 5x - y + 3z &= 0 \end{aligned}$$

$$40. \begin{aligned} 2x + z &= 1 \\ 5y - 3z &= 2 \\ 6x + 20y - 9z &= 11 \end{aligned}$$

$$41. \begin{aligned} x - y + 2z &= -4 \\ 3x + y - 4z &= -6 \\ 2x + 3y - 4z &= 4 \end{aligned}$$

$$42. \begin{aligned} x + y + z &= -3 \\ 4x + y - 3z &= 11 \\ 2x - 3y + 2z &= 9 \end{aligned}$$

$$\begin{aligned} & x + 2y + 6z = 5 \\ 43. \quad & -x + y - 2z = 3 \\ & x - 4y - 2z = 1 \end{aligned}$$

$$\begin{aligned} & x + 2y - 2z = 4 \\ 44. \quad & 2x + 5y - 7z = 5 \\ & 3x + 7y - 9z = 10 \end{aligned}$$

$$\begin{aligned} & 2x + y + 3z = 1 \\ 45. \quad & 2x + 6y + 8z = 3 \\ & 6x + 8y + 18z = 5 \end{aligned}$$

$$\begin{aligned} & 4x + y - 3z = 11 \\ 46. \quad & 2x - 3y + 2z = 9 \\ & x + y + z = -3 \end{aligned}$$

$$\begin{aligned} & 2x - 5y + z = 10 \\ 47. \quad & -x + 3y = -4 \\ & x - 2y + 2z = 9 \end{aligned}$$

$$\begin{aligned} & 2x + 13y + 6z = 1 \\ 48. \quad & x + 3z = 7 \\ & 2x + 10y + 8z = 8 \end{aligned}$$

$$\begin{aligned} & 3x - 2y + 3z = -4 \\ 49. \quad & 2x + y - 3z = 2 \\ & 3x + 4y + 5z = 8 \end{aligned}$$

$$\begin{aligned} & 3x - 3y + 4z = 6 \\ 50. \quad & 4x - 5y + 2z = 10 \\ & x - 2y + 3z = 4 \end{aligned}$$

$$\begin{aligned} & 3x - y + 2z = 2 \\ 51. \quad & 4x + 2y - 7z = 0 \\ & 2x + 3y - 5z = 7 \end{aligned}$$