13.1 2X2 Systems of Equations

In this section we want to spend some time reviewing systems of equations.

Recall there are two basic techniques we use for solving a system of equations: Elimination and Substitution.

We will start with the elimination method. The key idea in the elimination method is to eliminate one of the two variables by making the coefficients opposites on the two equations and adding the equations together. We will illustrate this method in the following example.

Example 1:

Solve using the elimination method.

a. $\begin{array}{l} 3x+2y=3 \\ 9x-8y=-2 \end{array}$ b. $\begin{array}{l} 5x-9y=7 \\ 7y-3x=-5 \end{array}$

Solution:

a. So to use the elimination method to solve, we need to start by determining which variable we would like eliminate. Lets choose the y variable since the signs are already opposite. Next we will need to multiply the top equation by a 4 to make the coefficients opposites. Then we simply add equations and solve. We get

$$\begin{array}{rr} 4(3x+2y)=4(3) \Rightarrow & 12x+8y=12 \\ 9x-8y=-2 \qquad & \underline{9x-8y=-2} \\ & 21x \qquad =10 \\ & x=\tfrac{10}{21} \end{array}$$

So now we need the y value of the solution. So we can simply substitute the x value into one of the original equations and solve for y. Let's use the first equation. We get

$$\begin{aligned} 3\left(\tfrac{10}{21}\right)+2y&=3 \\ \tfrac{10}{7}+2y&=3 \qquad \text{Clear fractions by multiplying by 7} \\ 10+14y&=21 \\ 14y&=11 \\ y&=\tfrac{11}{14} \end{aligned}$$

Since the solution to a system is an ordered pair we write the solution is $\left(\frac{10}{21}, \frac{11}{21}\right)$.

b. This time we need to start by putting the equations in standard form. Then we will proceed as above.

$$\begin{array}{rl} 5x-9y=7 \qquad & 5x-9y=7 \\ 7y-3x=-5 \Rightarrow & -3x+7y=-5 \end{array}$$

We will choose to eliminate the x's

$$\begin{array}{rr} 3(5x-9y)=3(7) \quad \Rightarrow & 15x-27y=21 \\ 5(-3x+7y)=5(-5) \Rightarrow & \underline{-15x+35y=-25} \\ & 8y=-4 \\ & y=-\tfrac{1}{2} \end{array}$$

Now we solve for x. We use the second equation this time. We get

$$\begin{aligned} 7\left(-\tfrac{1}{2}\right) - 3x &= -5 \\ -\tfrac{7}{2} - 3x &= -5 \\ -7 - 6x &= -10 \\ -6x &= -3 \\ x &= \tfrac{1}{2} \end{aligned}$$

So the solution is $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Next we will review the substitution method. The key idea in the substitution method is to solve one of the equations for one of the variables and then substitute that expression into the other equation. We illustrate this in the following example.

Example 2:

Solve using the substitution method.

a. $\begin{aligned} x + 2y &= 8 \\ x &= 4 - 3y \end{aligned}$ b. $\begin{aligned} 5x + 6y &= 14 \\ -3y + x &= 7 \end{aligned}$

Solution:

a. To use the substitution method we need to solve one equation for one variable. In this case, the bottom equation is already solved for x. Next, we substitute that expression into the other equation and solve for the remaining variable. We proceed as follows

$$\begin{aligned} &x + 2y = 8 \\ &\qquad \Rightarrow (4 - 3y) + 2y = 8 \\ &x = 4 - 3y \end{aligned}$$

$$\begin{aligned} (4 - 3y) + 2y &= 8 \\ 4 - 3y + 2y &= 8 \\ 4 - y &= 8 \\ -y &= 4 \\ y &= -4 \end{aligned}$$

Now we simply substitute this value back into the original equation for x, and solve. We get

$$\begin{aligned} x &= 4 - 3(-4) \\ &= 4 + 12 \\ &= 16 \end{aligned}$$

So our solution is $(16, -4)$.

b. So we need to start by solving one of the equation for one of the variables. We generally choose the easiest variable to get to. In this case that is the x on the bottom equation. So we solve for it. We get

$$\begin{aligned} -3y + x &= 7 \\ x &= 3y + 7 \end{aligned}$$

Now substitute as before and solve.

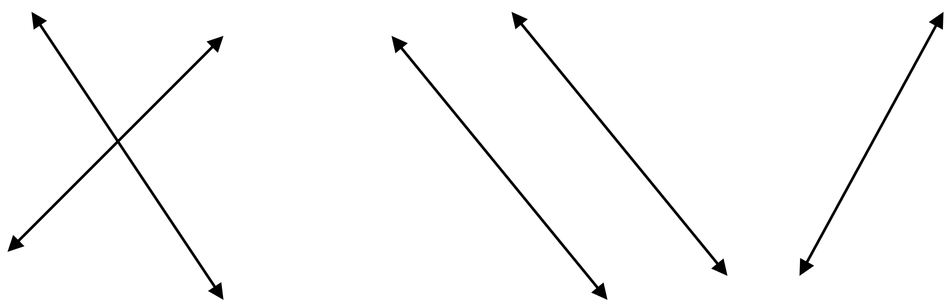
$$\begin{aligned} 5(3y+7)+6y &= 14 \\ 15y+35+6y &= 14 \\ 21y+35 &= 14 \\ 21y &= -21 \\ y &= -1 \end{aligned}$$

Again, substitute this value into the expression we obtained for x. We get

$$\begin{aligned} x &= 3(-1)+7 \\ &= -3+7 \\ &= 4 \end{aligned}$$

So our solution is $(4, -1)$.

Now, recall that graphically, the solution to a system of equations is the points of intersection of the graphs of the equations. So since, in this section, we are dealing only with lines, there are only three ways that they can intersect, that is, there are only three different possibilities for types of solutions to these systems.



One intersection point = One solution

No intersection point = No solution

Every point intersects = Infinite solutions

In order to determine if we have one of the two last cases we merely need to try to solve and interpret what we find.

<u>Example 3</u>:

Solve the following using any method.

a. $$\begin{aligned} \frac{x}{2}+\frac{y}{2} &= -1 \\ -3x-3y &= 6 \end{aligned}$$

b. $$\begin{aligned} -0.4x+0.7y &= 1.3 \\ 0.6x-1.05y &= 0.75 \end{aligned}$$

Solution:

a. First we will clear the fractions to evaluate which method of solving is best. So the first equation will be

$$\begin{aligned} 2\left(\frac{x}{2}+\frac{y}{2}\right) &= 2(-1) \\ x+y &= -2 \end{aligned}$$

Now, lets us the elimination method to solve. We will eliminate the x's.

$$\begin{aligned} 3(x+y) &= 3(-2) \\ -3x-3y &= 6 \end{aligned} \Rightarrow \begin{aligned} 3x+3y &= -6 \\ \underline{-3x-3y} &\underline{= 6} \\ 0 &= 0 \end{aligned}$$

Since all the variables eliminated we must have a special case. Since 0 = 0 for all values of x and y, we have the infinite solutions case. We simply say that the system has infinite solutions.

b. Again lets use the elimination method for solving. Again we will eliminate the x. We multiply the top equation by 0.6 and the bottom equation by 0.4. We get

$$\begin{aligned} 0.6(-0.4x+0.7y) &= 0.6(1.3) \\ 0.4(0.6x-1.05y) &= 0.4(0.75) \end{aligned} \Rightarrow \begin{aligned} -0.24x+0.42y &= 0.78 \\ \underline{0.24x-.42y} &\underline{= 0.3} \\ 0 &= 1.08 \end{aligned}$$

Once again we have eliminated all the variables. In this case however we are left with a statement that is not true. Since that is the case we have no solution.

Of course there are many very useful applications of systems of equations. We will concentrate on just a few different types: the mixture problem, the investment problem, the moving object problem.

We have reviewed the basic moving object problem in section R.5. The difference here is that we can now use two different variables to solve.

Example 4:

A motorboat traveling with the current went 112 km in 4 h. Against the current, the boat could go only 80 km in the same amount of time. Find the rate of the boat in calm water and the rate of the current.

Solution:
First we need to set up our variables. Lets let *r* be the rate of the boat in calm water and *c* be the rate of the current. That means that the total rate of the boat with the current would be *r* + *c* and the total rate against the current would be *r* – *c*. So, like we did in section R.5 we will set up a table to summarize our data and construct a system of equations.

	r	$\cdot\ t$	$= d$
With current	*r + c*	4	112
Against current	*r - c*	4	80

Now using the formula we get the system $\begin{aligned}(r+c)4 &= 112 \\ (r-c)4 &= 80\end{aligned}$. We need only solve the system. Lets use elimination. We will start by divide the each equation by 4 on both sides to simplify the system.

$$\frac{(r+c)4}{4} = \frac{112}{4}$$
$$\frac{(r-c)4}{4} = \frac{80}{4}$$

$$r + c = 28$$
$$\underline{r - c = 20}$$
$$2r \quad = 48$$
$$r = 24$$

So the rate of the boat is 25 km/hour. To find the rate of the current, *c*, we simply substitute it back into one of the equations and solve. We get

$$24 + c = 28$$
$$c = 4$$

So the current is 4 km/hour.

For the investment problem we simply use the formula $I = P \cdot r \cdot t$. Since our time will very often be one year, we can really just use $I = P \cdot r$. We use a table just like we will with the mixture problems.

There are two major types of mixture problems: Total Value and Total Quantity. The way we tell the difference is by the rate we are given. If the problem gives us a monetary rate (like $5 per pound) then we have a total value problem. If we have a percentage rate then we have a total quantity problem. We use the following formulas in conjunction with a table to solve:
Amount x Rate = Total Value (or simply $A \cdot r = TV$) or Amount x Rate = Quantity (or simply $A \cdot r = Q$). We will illustrate each type.

<u>Example 5</u>:

At the theatre, Jon buys 2 boxes of popcorn and 3 soft drinks for $6.05. One box of popcorn and one soft drink would cost $2.60. What is the cost of a box of popcorn?

Solution:
Since we are dealing with money here we use the formula $A \cdot r = TV$. Again, we will make a table for our data. We will let the cost of one box of popcorn be *p* and cost for one soft drink be *d*. We fill in the table with two different situations, the purchase that Jon makes and a purchase that would be only one of each. We get the following.

		A ·	r =	TV
Jon's purchase	popcorn	2	*p*	2*p*
	soft drinks	3	*d*	3*d*
One of each	popcorn	1	*p*	*p*
	soft drinks	1	*d*	*d*

The box in the top right is represents the total value of Jon's purchase and the box in the bottom right is the total value of a purchase of one or each. Therefore we get the system

$$2p + 3d = 6.05$$
$$p + d = 2.60$$

Lets solve this by substitution. Solving the bottom for *p* gives $p = 2.6 - d$. Substituting we get

$$2(2.6 - d) + 3d = 6.05$$
$$5.2 - 2d + 3d = 6.05$$
$$d = 0.85$$

So soft drinks are $0.85 each. Now we can solve for cost of popcorn. Substituting we get

$$p = 2.6 - 0.85 = 1.75$$

Thus popcorn costs $1.75 each.

Example 6:

A chemist wishes to make 2000 liters of a 3.5% acid solution by mixing a 2% solution with a 5.5% solution. How many liters of each solution are necessary?

Solution:
This time we will need to use the formula $A \cdot r = Q$ since we are given percentage rates. Also, we will need to use three rows, one for the 5.5% solution, one for the 2% solution and one for the final solution. Lets call the amount of 2% solution *x* and the amount of 5.5% solution *y*. So we fill in the table as follows.

	A	$\cdot\ r$	$= Q$
2% solution	*x*	0.02	0.02*x*
5.5% solution	*y*	0.055	0.055*y*
Final solution	2000	0.035	(2000)(0.035)

The way that this type of mixture problem works is the final amount must always equal the sum of the amounts of the two being mixed, and the final quantity must always equal the sum of the two quantities. So that being said we get the system

$$x + y = 2000$$
$$0.02x + 0.055y = 70$$

We will solve this system by the elimination method.

$$\begin{aligned} -0.02(x+y) &= -0.02(2000) \\ 0.02x + 0.055y &= 70 \end{aligned} \Rightarrow \begin{aligned} -0.02x - 0.02y &= -40 \\ 0.02x + 0.055y &= 70 \\ \hline 0.035y &= 30 \\ y &\approx 857.1 \end{aligned}$$

Therefore, $x \approx 1{,}142.9$.
So we need about 1,142.9 liters of 2% solution and 857.1 liters of 5.5% solution.

13.1 Exercises

Solve the following using the elimination method.

1. $\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}$

2. $\begin{aligned} x - 2y &= 0 \\ 2x - y &= 6 \end{aligned}$

3. $\begin{aligned} x - 2y &= 5 \\ -3x + 6y &= 4 \end{aligned}$

4. $\begin{aligned} 5x + y &= 4 \\ 5x - 3y &= 8 \end{aligned}$

5. $\begin{aligned} 3x - y &= 2 \\ 6x - 2y &= 4 \end{aligned}$

6. $\begin{aligned} 2x + 3y &= 2 \\ 3x - 2y &= 3 \end{aligned}$

7. $\begin{aligned} x + 10y &= -7 \\ -2x + 5y &= 4 \end{aligned}$

8. $\begin{aligned} 5x + 3y &= 4 \\ 2x - y &= 5 \end{aligned}$

9. $\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y &= \tfrac{25}{6} \\ x - y &= 5 \end{aligned}$

10. $\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{4}y &= \tfrac{7}{4} \\ \tfrac{2}{3}x - \tfrac{1}{3}y &= \tfrac{13}{3} \end{aligned}$

11. $\begin{aligned} 0.2x + 0.3y &= 1 \\ 0.3x - 0.2y &= 4 \end{aligned}$

12. $\begin{aligned} 1.5x - 3 &= -2y \\ 0.3x + 0.4y &= 0.6 \end{aligned}$

Solve the following using the substitution method.

13. $\begin{aligned} x+y&=4 \\ x-y&=2 \end{aligned}$

14. $\begin{aligned} x-2y&=0 \\ 2x-y&=6 \end{aligned}$

15. $\begin{aligned} x-2y&=5 \\ -3x+6y&=4 \end{aligned}$

16. $\begin{aligned} 5x+y&=4 \\ 5x-3y&=8 \end{aligned}$

17. $\begin{aligned} 3x-y&=2 \\ 6x-2y&=4 \end{aligned}$

18. $\begin{aligned} 2x+3y&=2 \\ 3x-2y&=3 \end{aligned}$

19. $\begin{aligned} x+10y&=-7 \\ -2x+5y&=4 \end{aligned}$

20. $\begin{aligned} 5x+3y&=4 \\ 2x-y&=5 \end{aligned}$

21. $\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y&=\tfrac{25}{6} \\ x-y&=5 \end{aligned}$

22. $\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{4}y&=\tfrac{7}{4} \\ \tfrac{2}{3}x-\tfrac{1}{3}y&=\tfrac{13}{3} \end{aligned}$

23. $\begin{aligned} 0.2x+0.3y&=1 \\ 0.3x-0.2y&=4 \end{aligned}$

24. $\begin{aligned} 1.5x-3&=-2y \\ 0.3x+0.4y&=0.6 \end{aligned}$

Solve using any method.

25. $\begin{aligned} x+y&=-1 \\ y&=x+1 \end{aligned}$

26. $\begin{aligned} 3x-y&=6 \\ x+3y&=2 \end{aligned}$

27. $\begin{aligned} y&=2x+11 \\ y-5x&=-19 \end{aligned}$

28. $\begin{aligned} y&=x-5 \\ y+2x&=4 \end{aligned}$

29. $\begin{aligned} x+y&=1 \\ 3x-y&=-5 \end{aligned}$

30. $\begin{aligned} y&=-\tfrac{1}{2}x \\ y-x&=3 \end{aligned}$

31. $\begin{aligned} 3x+2y&=6 \\ 2y&=-6 \end{aligned}$

32. $\begin{aligned} y&=3x \\ 4y-12x&=8 \end{aligned}$

33. $\begin{aligned} y&=-6x \\ y&=x-5 \end{aligned}$

34. $\begin{aligned} y&=x-3 \\ 3x+2y&=9 \end{aligned}$

35. $\begin{aligned} 5x-3y&=2 \\ 2x-7y&=-1 \end{aligned}$

36. $\begin{aligned} 2x-5y&=1 \\ 4x+3y&=0 \end{aligned}$

37. $\begin{aligned} 10x-5y&=7 \\ 2x-y&=4 \end{aligned}$

38. $\begin{aligned} 5x-2y&=3 \\ x&=y-4 \end{aligned}$

39. $\begin{aligned} 8x-2y&=-5 \\ 3x+4y&=1 \end{aligned}$

40. $\begin{aligned} 3x-4y&=0 \\ 4y-3x&=0 \end{aligned}$

41. $\begin{aligned} \tfrac{3}{4}x+\tfrac{1}{3}y&=-\tfrac{1}{2} \\ \tfrac{1}{2}x-\tfrac{5}{6}y&=-\tfrac{7}{2} \end{aligned}$

42. $\begin{aligned} \tfrac{2}{5}x-\tfrac{1}{3}y&=1 \\ \tfrac{3}{5}x+\tfrac{2}{3}y&=5 \end{aligned}$

43. $\begin{aligned} 0.5x+0.4y&=0 \\ 0.3x+0.7y&=0 \end{aligned}$

44. $\begin{aligned} 0.3x+0.7y&=1.6 \\ 0.4x-0.3y&=0.9 \end{aligned}$

Set up a system for the following problems and solve using any method.

45. A boat travels 36 miles in 4 hours upstream. In the same amount of time the boat can travel 48 miles downstream. Find the rate of the current and the rate of the boat in still water.

46. A 24-foot rope is cut in two pieces so that one piece is twice as long as the other. How long is each piece?

47. How many grams of silver that is 60% pure must be mixed together with silver that is 35% pure to obtain a mixture of 90 grams of silver that is 45% pure?

48. At a barbecue, there were 250 dinners served. Children's plates were $1.50 each and adult's plates were $2.00 each. If the total amount of money collected for dinners at the barbecue was $441, how many of each type of plate was served?

49. A store sells 45 shirts, one kind at $8.50 and the other at $9.75. In all, $398.75 was taken in. How many of each type were sold?

50. A store buyer purchased 12 regular calculators and 5 graphing calculators for a total cost of $370.75. A second purchase, at the same prices, included 10 regular calculators and 3 graphing calculators for a total of $246.25. Find the cost for each type of calculator.

51. Flying with the wind, a plane flew 1000 miles in 4 hours. Flying against the wind, the plane could fly only 500 miles in the same amount of time. Find the rate of the plane in calm air and the rate of the wind.

52. A plane traveling with the wind flew 3625 miles in 6.25 hours. Against the wind, the plane required 7.25 hours to go the same distance. Find the rate of the plane in calm air and the rate of the wind.

53. A rowing team rowing with the current traveled 45 mi in 3 h. Against the current, the team rowed 27 mi in 3 h. Find the rate of the rowing team in calm water and the rate of the current.

54. How many grams of pure acid must be added to 20% acid to make 96 grams of solution that is 50% acid?

55. A chemist has some 8% hydrogen peroxide solution and some 3% hydrogen peroxide solution. How many milliliters of the 8% solution should be used to make 500 milliliters of solution that is 4.2% hydrogen peroxide?

56. How many grams of 10% pure gold must be mixed with 20% pure gold to make 25 grams of 16% pure gold?

57. A total of $6000 is deposited into two simple interest accounts. One account has an interest rate of 7.5% and the other has a rate of 11.4%. How much should be invested in the 11.4% account to earn a total interest of $528?

58. A total of $32,000 is invested to provide retirement income. Part of the $32,000 is invested in an account paying 9% interest. How much should be invested into an 11% interest account so that the total income is $3280?

59. A total of $18,000 is invested into two simple interest accounts. One account has an interest rate of 8% while the other account has a rate of 10%. How much must be invested in each account so that both accounts earn the same amount of interest?

60. Find the cost per kilogram of a grated cheese mixture made from 6 kg of cheese that costs $10.50 per kilogram and 15 kg of cheese that costs $7.00 per kilogram.

61. How many grams of pure acid must be added to 60 g of a 20% acid solution to make a solution that is 50% acid?

62. A researcher mixes 500 g of 12% aluminum alloy with 100 g of a 40% aluminum alloy. What is the percent concentration of the resulting alloy?

63. A total of $9500 is deposited into two simple interest accounts. On one account the annual simple interest rate is 10%; on the second account the annual simple interest rate is 11%. How much should be invested in the 11% account so that the total interest earned is $1005?

64. An investment of $3000 is made into a 10% simple interest account. How much additional money must be deposited into a 12.5% simple interest account so that the total interest earned on both accounts is 11% of the total investment?

65. A jet plane traveling at 570 mph overtakes a propeller-driven plane that has a 2.8 hour head start. The propeller-driven plane is traveling at a rate of 150 mph. How far from the starting point does the jet overtake the propeller-driven plane?

66. A student pilot flies to a city at an average speed of 100 mph and then returns at an average speed of 150 mph. Find the total distance between the two cities if the total flying time was 5 hours.