12.6 Further Topics in Conic Sections

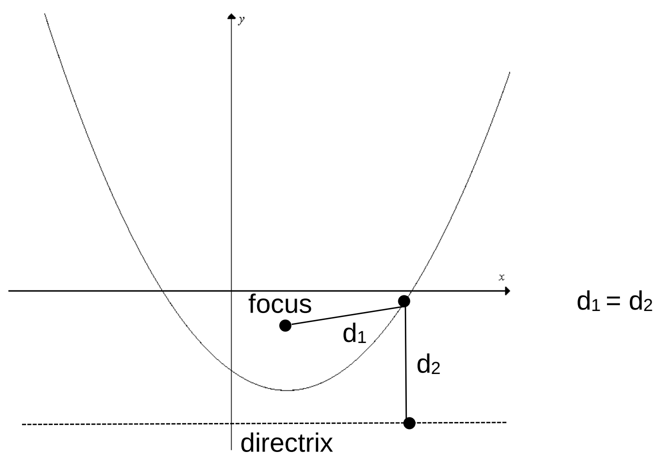
Now that we have a basic idea of how that primary conic sections work, we want to spend this section dealing with some of the finer details of conic sections. Namely, we want to take a look at some very special points, lines and values regarding conics.

Let's start with some values for the parabola. Namely, we want to look at what are called the **focus** and **directrix** of the parabola.

First of all, as it turns out, the parabola can be generated much more geometrically then just plugging points into an equation.

We begin by redefining the parabola.

Definition: **Parabola**- the set of all points equally distant from a fixed line and a fixed point, not on the line. The fixed line is called the **directrix** and the fixed point is called the **focus**.



With this new definition, it can be shown that we have a new standard form of the parabola. The proof this requires the use of the distance formula applied to the definition above. The proof can be found in most Precalculus texts and will be omitted here for the sake of brevity.

Standard Form of a Parabola

The standard forms of the parabola with vertex (h, k) are

$(x-h)^2 = 4p(y-k)$, for vertical axis of symmetry of $x = h$, and

$(y-k)^2 = 4p(x-h)$, for horizontal axis of symmetry of $y = k$

The value of *p* is the distance from the vertex to the focus as well as the shortest distance from the vertex to the directrix. Recall, the vertex is the maximum, minimum, extreme left or extreme right value of the parabola, depending on which type of parabola we have.

Let's see how this all works with an example.

Example 1:

Graph. Find the vertex, focus and directrix.

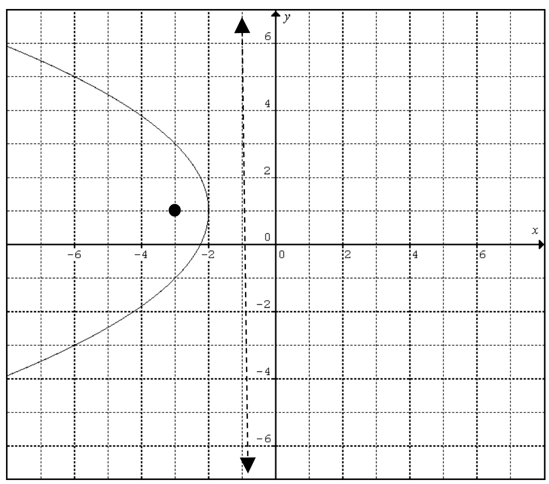
a. $(y-1)^2 = -4(x+2)$ b. $x^2 + 2x - 8y - 15 = 0$

Solution:

a. First, notice that the parabola is already in standard form. Therefore, the vertex is (-2, 1). Now we simply need to find the value of *p* in order to determine where the focus and directrix will be.

According to the definition above, 4*p* = -4. Therefore, *p* = -1. Since *p* is negative, this means that the parabola will open to the left (the negative direction on the horizontal axis) and the focus will be 1 unit to the left of the vertex since the focus is always on the inside of the parabola. The directirx is always behind the parabola and will be 1 unit back from the vertex in this case.

So we get the graph



Therefore, the focus is (-3, 1) and the directrix is *x* = -1.

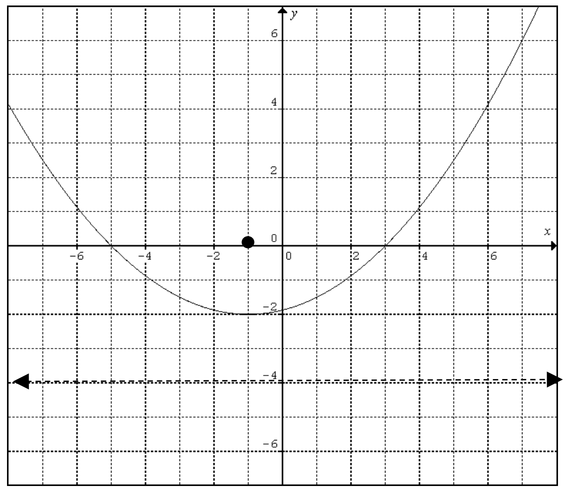
b. The first this we need to do is get the parabola into standard form so that we can determine the vertex and the value of *p*. We use completing the square as we did before. We get

$$x^2 + 2x - 8y - 15 = 0$$
$$x^2 + 2x = 8y + 15$$
$$x^2 + 2x + 1 = 8y + 15 + 1$$
$$(x+1)^2 = 8y + 16$$
$$(x+1)^2 = 8(y+2)$$

So, the vertex is (-1, -2) and basic solving gives *p* = 2.

Since the *p* value is positive and the axis is vertical, the focus will be up, 2 units from the vertex, and the directrix will be below the parabola by 2 units.

So our graph is

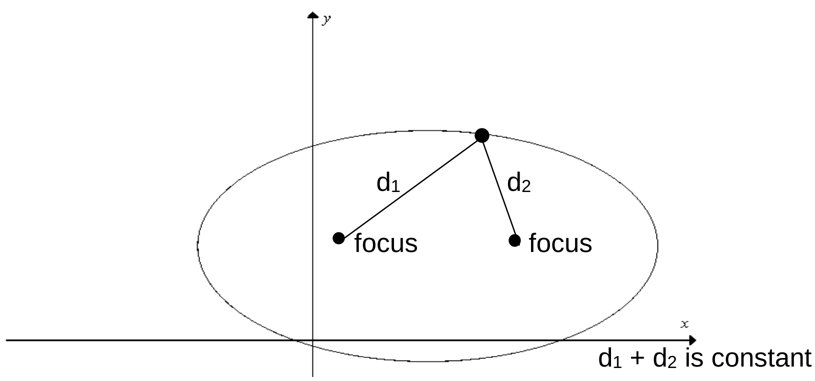


Therefore, our focus is (-2, 0) and the directrix is y = -4.

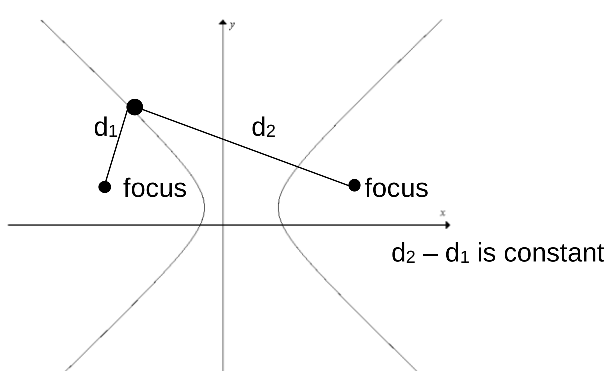
Next we want to look at some interesting points for the ellipse and the hyperbola. These are called the **foci** (plural for focus). We also want to talk about a value called the **eccentricity**.

As was the case with the parabola, we can generate the ellipse and hyperbola much more geometrically.

Definition: **Ellipse**- the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a **focus**.



Definition: **Hyperbola**- the set of all points, the difference of whose distances from two fixed points is a positive constant. Each fixed point is called a **focus**.



In these cases, the definition does not alter the way the standard form looks. So, the standard form for the ellipse and hyperbola that we used earlier in the chapter are still the same.

However, we still need to be able to find the foci. To find the values for these foci, we simply need a couple of formulas. As with the parabola, these formulas are proven in most Precalculus texts and is omitted here.

We are also interested in finding a very special value, called the **eccentricity**, for these conics. The value for the eccentricity always describes the conic you are working with.

Foci and Eccentricity of the Ellipse and Hyperbola

1. The foci for the ellipse with standard form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ are *c* units away from the center, on the major axis, where $c^2 = a^2 - b^2$ for a horizontal axis and $c^2 = b^2 - a^2$ for a vertical axis.

   The eccentricity of the ellipse with horizontal major axis is $e = \frac{c}{a}$ and with vertical axis is $e = \frac{c}{b}$.

2. The foci for the hyperbola with standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ are *c* units away from the center, on the axis of symmetry, where $c^2 = a^2 + b^2$.

   The eccentricity of the hyperbola with horizontal axis is $e = \frac{c}{a}$ and with vertical axis is $e = \frac{c}{b}$.

So, what exactly is the eccentricity? The simple answer is that it's value will tell you exactly which of the conics you are working with as well as just how "ellipse like" or "circle like" your graphs are.

Here is how you can use the eccentricity.

- If $e = 0$, then the conic is a circle.
- If $0 < e < 1$, then the conic is an ellipse. The lower the value, the more circular the ellipse, the higher the value, the more elliptical the conic.
- If $e = 1$, then the conic is a parabola.
- If $e > 1$, then the conic is a hyperbola.

This can be very helpful when you are graphing your conic to, yet again, verify that all of the information fits together like it should.

Let's put it all together with the next couple of examples.

Example 2:

Graph. Give the center, vertices, foci and eccentricity.

a. $\frac{(x-1)^2}{4} + \frac{(y+4)^2}{9} = 1$

b. $x^2 + 16y^2 - 160y + 384 = 0$

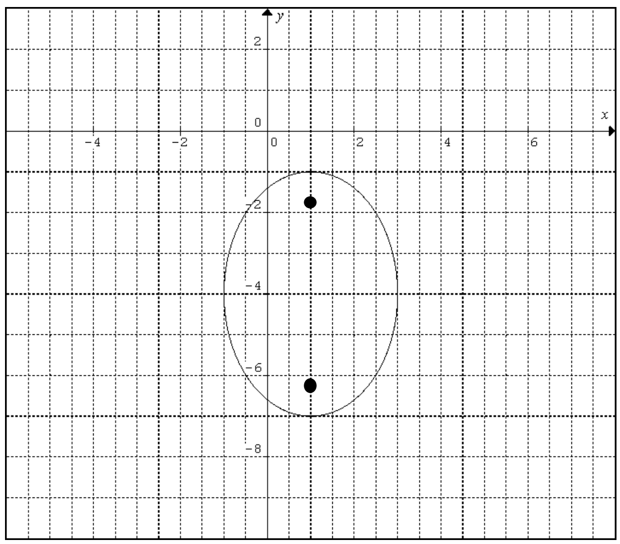
Solution:

a. First of all, we see that the ellipse given is already in standard form. This means that the center is (1, -4) and that a = 2 and b = 3.

So all we need to find is the value of c to be able to produce the graph as well as determine e. Once we have the graph, we can simply read off the values of the vertex and foci. We get c by using the formula above. We have

$$\begin{aligned} c^2 &= b^2 - a^2 \\ c^2 &= 9 - 4 \\ c^2 &= 5 \\ c &= \sqrt{5} \end{aligned}$$

So the foci are $\sqrt{5} \approx 2.2$ units from the center on the major axis.
So we have



So our vertices are (1, -1) and (1, 7). Since the foci are on the major axis and 2.2 units away from the center, the foci must be (1, -4+2.2) and (1, -4-2.2). Which gives (1, -1.8) and (1, -6.2).

Lastly, we need the eccentricity, e. This value is just $e = \frac{c}{b} = \frac{\sqrt{5}}{3} \approx 0.745.$

b. Clearly, the first thing we need to do is get the equation into standard form. As we did before, we use completing the square.

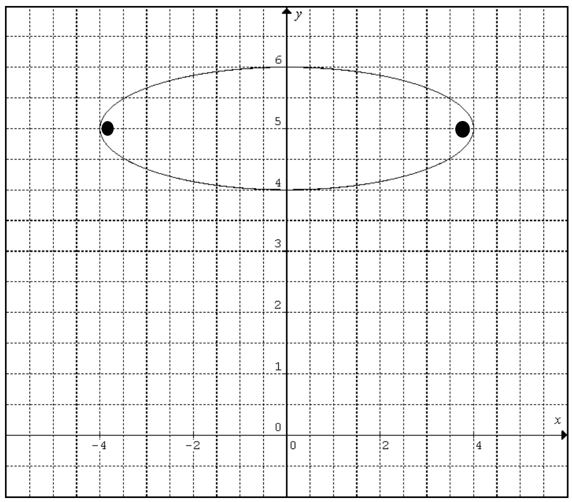
$$\begin{aligned} x^2 + 16y^2 - 160y + 384 &= 0 \\ x^2 + 16(y^2 - 10y) &= -384 \\ x^2 + 16\left(y^2 - 10y + \left(\frac{1}{2}\cdot 10\right)^2\right) &= -384 + 16\left(\frac{1}{2}\cdot 10\right)^2 \\ x^2 + 16(y-5)^2 &= -384 + 400 \\ x^2 + 16(y-5)^2 &= 16 \\ \frac{x^2}{16} + \frac{(y-5)^2}{1} &= 1 \end{aligned}$$

So, the center is (0, 5), a = 4 and b = 1.

Therefore, we get *c* as follows

$$c^2 = a^2 - b^2$$
$$c^2 = 4^2 - 1^2$$
$$c^2 = 15$$
$$c = \sqrt{15} \approx 3.9$$

So, the foci will be 3.9 units from the center along the major axis (horizontal). This gives us the graph



Our vertices are clearly (4, 5) and (-4, 5). Our foci are (3.9, 5) and (-3.9, 5). Lastly, we have $e = \frac{c}{a} = \frac{\sqrt{15}}{4} \approx 0.968.$

Example 3:

Graph. Give the center, vertices, foci and eccentricity.

a. $\frac{(x-5)^2}{9} - \frac{(y+1)^2}{25} = 1$

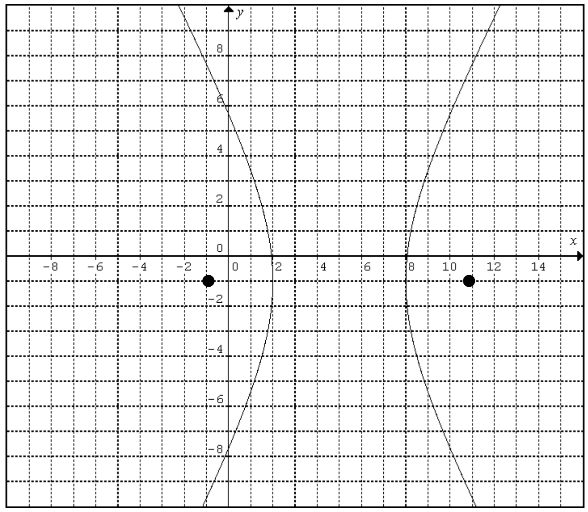
b. $y^2 - x^2 - 8y + 8x - 25 = 0$

Solution:

a. First, we notice that the hyperbola is already in standard form. So, we know that the center is (5, -1), *a* = 3 and *b* = 5. With these values we can find the value of *c* by simply plugging it into the formula. We get

$$c^2 = a^2 + b^2$$
$$c^2 = 3^2 + 5^2$$
$$c^2 = 34$$
$$c = \sqrt{34} \approx 5.8$$

So the foci are 5.8 units away from the center, along the axis (horizontal). Therefore, we get the following graph



From the graph we can clearly see that the vertices are (8, -1) and (2, -1). Also, the foci are $(5 \pm 5.8, -1)$. That is, (10.8, -1) and (-0.8, -1).

The last thing we need is the eccentricity.
Using the formula, we get $e = \frac{c}{a} = \frac{\sqrt{34}}{3} \approx 1.944.$

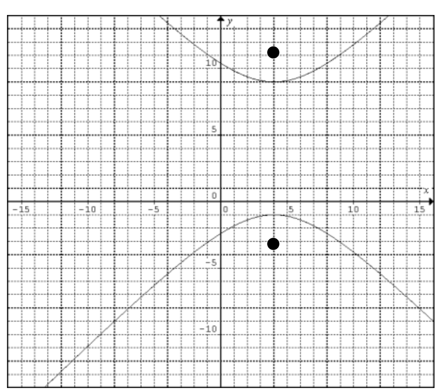
b. This time we need to start by using completing the square to get the equation into standard form.

$$y^2 - x^2 - 8y + 8x - 25 = 0$$
$$y^2 - 8y - x^2 + 8x = 25$$
$$(y^2 - 8y) - (x^2 - 8x) = 25$$
$$\left(y^2 - 8y + \left(\frac{1}{2} \cdot 8\right)^2\right) - \left(x^2 - 8x + \left(\frac{1}{2} \cdot 8\right)^2\right) = 25 + \left(\frac{1}{2} \cdot 8\right)^2 - \left(\frac{1}{2} \cdot 8\right)^2$$
$$(y-4)^2 - (x-4)^2 = 25$$
$$\frac{(y-4)^2}{25} - \frac{(x-4)^2}{25} = 1$$

So, the center is (4, 4) and *a* = *b* = 5. Calculating *c* we get

$$c^2 = a^2 + b^2$$
$$c^2 = 5^2 + 5^2$$
$$c^2 = 50$$
$$c = \sqrt{50} \approx 7.1$$

Our graph, then, is



Therefore, our vertices are (4, 9) and (4, -1). Also, our foci are $(4, 4 \pm 7.1)$. That is, (4, 11.1) and (4, -3.1).

Finally, $e = \frac{c}{b} = \frac{\sqrt{50}}{4} \approx 1.768.$

Lastly, we need to be able to reconstruct the equation of a conic based on these important values. Let's see how to do this with the following example.

<u>Example 4</u>:

Find the standard form of the conic section under the given conditions.

a. the parabola with vertex of (-1, 3) and focus at (0, 3).
b. the conic with foci at $(0, \pm 1)$ and vertices at (0 ± 4).
c. the conic with eccentricity of 2 and foci of (5, 2) and (1, 2).

Solution:

a. First, since we know we are working with a parabola, we need to begin by determining if it opens up, down, left or right. Since we know that the focus is always on the interior of the parabola and the focus in this case is directly to the right, we must be working with a parabola which is opening to the right.

   This means we are looking for the parabola of form $(y-k)^2 = 4p(x-h)$. Also, since we already know the vertex, we can insert the values directly in. This gives us

$$(y-3)^2 = 4p(x+1)$$

   All we need now is to find the value of *p*. Recall, *p* is the distance from the vertex to the focus. The distance here is 1 unit. Therefore, *p* = 1. So, our standard form of this parabola is

$$(y-k)^2 = 4(x-h)$$

b. The first thing we notice is that we are not told which conic we are actually dealing with here. However, the first thing we can do is find the center. The center must be the midpoint of the vertices, since the vertices are always symmetrically away from the center, along the axis. We can use the midpoint formula if we would like, however, in this case, clearly we have a center of (0, 0).

   Also, the foci are equally spaced from the center and since the foci are closer than the vertices to the center, we must be working with an ellipse (since the foci are always on the inside of the ellipse).

   So we are looking to find an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   We know *b* = 4 because the axis must be vertical since the vertices and foci are vertical and c = 1. This just means we only need to find *a*. This can be done by just solving for *a* in the equation $c^2 = b^2 - a^2$. We get

$$1^2 = 4^2 - a^2$$
$$-15 = -a^2$$
$$a = \sqrt{15}$$

Putting it all together we get the equation

$$\frac{x^2}{15}+\frac{y^2}{16}=1$$

c. First, since we have an eccentricity that is larger than 1, we know we are working with a hyperbola. Also, since the foci are horizontally oriented we must be working on a hyperbola of the form

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Now, we need to find the center. Since the foci are always symmetric about the center, the midpoint of the foci is the center. Using the midpoint formula or simply graphing the foci we can easily get a center of (3, 2).

We can also find the value of *c* by recalling that the foci are *c* units from the center. This means that *c* = 2. The reason we need this value is since we were given the value of *e* and we know we have a horizontal hyperbola, we know $e=\frac{c}{a}$. This will allow us to find the value of *a*. Solving we get

$$e=\frac{c}{a}$$

$$2=\frac{2}{a}$$

$$a=1$$

So, at this point we have the equation

$$\frac{(x-3)^2}{1}-\frac{(y-2)^2}{b^2}=1$$

All we need to find, therefore, is the value of *b*. Since we have *a* and *c*, all we need to do is solve $c^2=a^2+b^2$ for *b*. We get

$$c^2=a^2+b^2$$
$$2^2=1^2+b^2$$
$$4=1+b^2$$
$$b^2=3$$
$$b=\sqrt{3}$$

So our equation is

$$(x-3)^2-\frac{(y-2)^2}{3}=1$$

12.6 Exercises

Graph. Find center, vertices, foci and directrix, if applicable.

1. $(x-3)^2=4(y-1)$

2. $(x+2)^2=8(y+2)$

3. $(y+2)^2=-2(x-3)$

4. $(y-1)^2=-4(x+5)$

5. $y^2-6y-4x+17=0$

6. $y^2+2y+8x+17=0$

7. $x^2-8x-y+18=0$

8. $x^2+6y+18=0$

9. $2y^2 - 4y - x + 5 = 0$

10. $\frac{1}{4}y^2 - y - x + 1 = 0$

11. $\frac{(x-5)^2}{25} + \frac{(y+1)^2}{9} = 1$

12. $\frac{(x+1)^2}{4} + \frac{(y+2)^2}{9} = 1$

13. $(x+3)^2 + 9y^2 = 9$

14. $4(x-2)^2 + (y+2)^2 = 4$

15. $3x^2 + 4y^2 - 6x + 16y + 7 = 0$

16. $16x^2 + 9y^2 + 64x - 54y + 1 = 0$

17. $4x^2 + 9y^2 - 8x - 54y + 49 = 0$

18. $x^2 + 16y^2 - 160y + 384 = 0$

19. $5x^2 + 3y^2 - 40x - 36y + 113 = 0$

20. $4x^2 + 9y^2 - 32x + 36y + 127 = 0$

21. $\frac{(x-5)^2}{25} - \frac{(y+1)^2}{9} = 1$

22. $\frac{(x+3)^2}{16} - \frac{(y-4)^2}{16} = 1$

23. $(y-2)^2 - 4(x-1)^2 = 4$

24. $9(y+1)^2 - 4(x+3)^2 = 36$

25. $9y^2 - 25x^2 - 54y + 200x - 544 = 0$

26. $4x^2 - y^2 - 32x + 2y + 67 = 0$

27. $x^2 - y^2 + 2y - 5 = 0$

28. $x^2 - y^2 + 7x - y + 8 = 0$

29. $y^2 - 25x^2 + 8y - 9 = 0$

30. $9x^2 - 16y^2 + 9x + 4y + 2 = 0$

Find the standard form of the conic section under the given conditions.

31. the parabola with vertex of (0, 0) and focus (0, 3)

32. the parabola with vertex of (1, 0) and directrix x = 3

33. the parabola with focus of (5, -3) and directrix x = -9

34. the parabola with vertex of (-2, 4) and focus (-2, 5)

35. the conic with foci at (-1, 0) and (1, 0) and with eccentricity $\frac{1}{3}$

36. the conic with foci at (0, -4) and (0, 4) and with eccentricity $\frac{1}{2}$

37. the conic with eccentricity of $\frac{\sqrt{5}}{3}$ and vertices of (4, 3) and (-2, 3)

38. the conic with eccentricity of $\frac{1}{4}$ and vertices of (4, 1) and (-4, 1)

39. the conic with foci of (4, 2) and (-2, 2) and vertices of (6, 2) and (-4, 2)

40. the conic with foci of (-3, 1) and (5, 1) and vertices of (-5, 1) and (6, 1)

41. the conic with foci of (4, 0) and (-4, 0) and vertices of (1, 0) and (-1, 0)

42. the conic with foci of (0, 5) and (0, -5) and vertices of (0, 3) and (0, -3)

43. the conic with eccentricity of $\frac{\sqrt{34}}{5}$ and vertices of (10, -1) and (0, -1)

44. the conic with eccentricity of $\frac{\sqrt{5}}{2}$ and vertices of (1, 4) and (1, 0)

45. the conic with eccentricity of $\sqrt{2}$ and foci of $(2 \pm 3\sqrt{2}, 1)$

46. the conic with eccentricity of 2 and foci of (1, 4) and (-1, 4)