

## 12.5 The Hyperbola

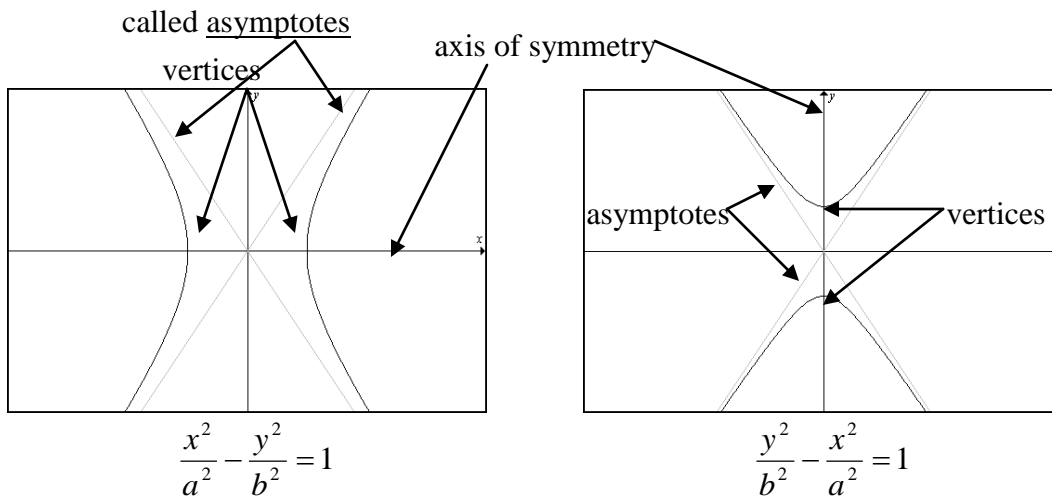
The last one of our conic sections is the hyperbola. We will again start with it centered at the origin before looking at the hyperbola in general.

### Standard Form of a Hyperbola Centered at (0, 0)

-The equation of a hyperbola where the axis of symmetry is the x-axis is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . The vertices are  $(a, 0)$  and  $(-a, 0)$ .

-The equation of a hyperbola where the axis of symmetry is the y-axis is  $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ . The vertices are  $(0, b)$  and  $(0, -b)$ .

This gives us two possible pictures



We use these asymptotes to draw the graph. As the graph continues, it gets closer and closer to the asymptotes but never crosses the asymptotes.

Now, how do we really graph these? We use something we call a Central Rectangle. Basically, a central rectangle is a rectangle that you make by using the  $a$  and  $b$  values in the same fashion that we did for graphing an ellipse. We go right and left  $a$  units (since the  $a^2$  is under the  $x$ ) and up and down  $b$  units (since  $b^2$  is under the  $y$ ) from the center. Then we draw the asymptotes diagonally across the rectangle. We then complete the graph by remembering the following fact: A hyperbola always surrounds the axis of the variable which had the positive values in the equation.

So if the  $x$  part was positive, the hyperbola will open in the  $x$  directions. If the  $y$  part is positive, the hyperbola will open in the  $y$  directions.

#### Example 1:

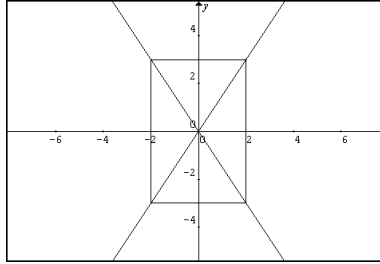
Graph the following.

a.  $\frac{x^2}{4} - \frac{y^2}{9} = 1$

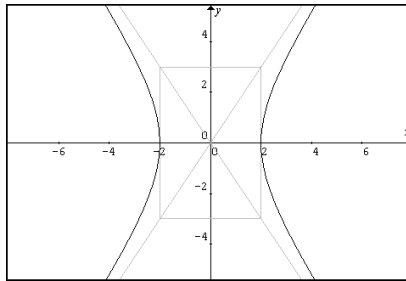
b.  $y^2 = x^2 + 4$

Solution:

- a. First we notice that the equation is in standard form with  $a = 2$  and  $b = 3$ . Also, since the  $x$  part is positive, we know that the axis of symmetry is the  $x$ -axis. So we make our central rectangle using the  $a$  and  $b$  as we did with the ellipse. Then draw the asymptotes diagonally across the rectangle as follows.



Now we simply draw the hyperbola so that its axis of symmetry is the  $x$ -axis and the graph moves towards the asymptotes. We also need to make sure that the graph actually touches the central rectangle. These points are our vertices. We get



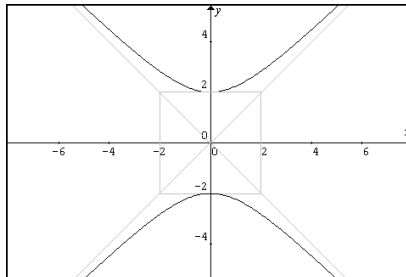
- b. This time we notice the equation is not in standard form. However, we can easily put it into standard form as follows

$$y^2 = x^2 + 4$$

$$y^2 - x^2 = 4$$

$$\frac{y^2}{4} - \frac{x^2}{4} = 1$$

So we see that both  $a = 2$  and  $b = 2$ . As above we will make our central rectangle and asymptotes and this time the axis of symmetry is the  $y$  axis, since the  $y$  part is positive. We get



Now we want to do the hyperbola in general.

## Standard Form of a Hyperbola

The equation of the hyperbola with center  $(h, k)$  are

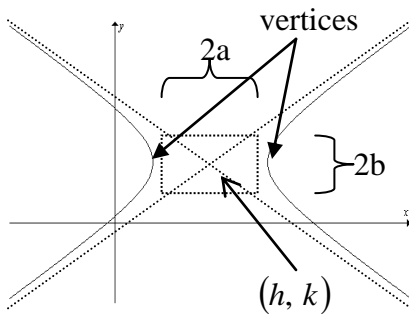
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ - horizontal axis of symmetry, } y = k ;$$

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \text{ - vertical axis of symmetry, } x = h .$$

The vertices are always on the axis of symmetry where the graph hits the central rectangle.

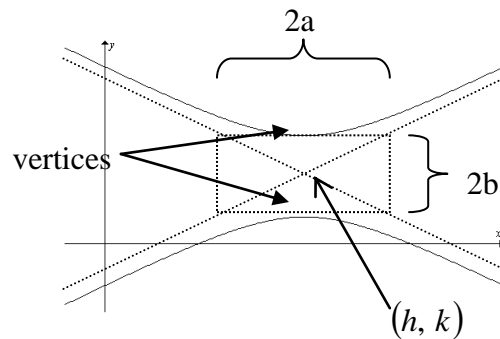
Notice that as usual the  $x$  part of the center is with the  $x$  variable and the  $y$  part is with the  $y$  variable.

Again we get two possible graphs.



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Horizontal axis



$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Vertical axis

We graph these the same way we did above, its just that now to make our central rectangle we have to start at our center and move  $a$  and  $b$  units from there.

### Example 2:

Graph the following. Find the center, vertices and  $x$ - and  $y$ -intercepts.

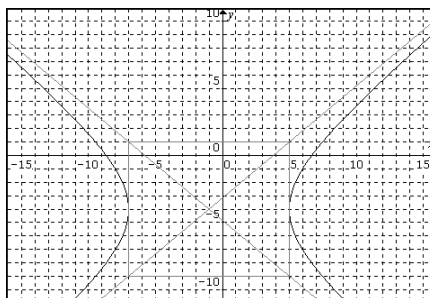
a.  $\frac{(x+1)^2}{36} - \frac{(y+4)^2}{25} = 1$

b.  $16y^2 - 9x^2 + 72x + 32y - 272 = 0$

Solution:

- a. First, we notice that the equation is already in standard form. Therefore, we can determine the center and our  $a$  and  $b$  values. Similar to the circle and ellipse we carefully extract the center and get  $(-1, -4)$ . Also, we can see that we have  $a = 6$  and  $b = 5$ . Now before we determine the vertices and intercepts its easiest to graph at this point and then we can simply read the vertices off the graph.

Again we will start with the central rectangle. We need to center our rectangle at the center of the hyperbola. We can then make the asymptotes as above and finish the graph as we did in the previous example remembering that since the  $x$  part is positive, the graph will open in the  $x$  directions (right and left). We get



Since the graph opens right and left, and the vertices are on the axis, we can just read the vertices off the graph. We get  $(4, -4)$  and  $(-7, -4)$ .

Now all we need is the intercepts. However, we can see that the graph does not hit the y axis and therefore should not have any y-intercepts. We can verify this by calculating them as we usually do. That is, for x-intercepts, let  $y = 0$  and for y-intercepts, let  $x = 0$ .

x-intercepts:

$$\frac{(x+1)^2}{36} - \frac{(0+4)^2}{25} = 1$$

$$25(x+1)^2 - 576 = 900$$

$$25(x+1)^2 = 1476$$

$$x = -1 \pm \sqrt{\frac{1476}{25}}$$

$$x \approx -8.7, 6.7$$

y-intercepts:

$$\frac{(0+1)^2}{36} - \frac{(y+4)^2}{25} = 1$$

$$25 - (y+4)^2 = 900$$

$$(y+4)^2 = -875$$

$$y = -4 \pm \sqrt{-875}$$

Since we have complex solutions for our y-intercepts we clearly have no y-intercepts as we had seen from the graph. The x-intercepts we found are also consistent with our graph.

- b. This time we have an equation that is not in standard form. So we must start by getting it into standard form by completing the square as we did in the previous sections. This time we need to be extra careful with the coefficients in front of the negative squared variable. We proceed as follows

$$16y^2 - 9x^2 + 72x + 32y - 272 = 0$$

$$16y^2 + 32x - 9x^2 + 72x = 272$$

$$16(y^2 + 2y) - 9(x^2 - 8x) = 272$$

$$16\left(y^2 + 2y + \left(\frac{1}{2} \cdot 2\right)^2\right) - 9\left(x^2 - 8x + \left(\frac{1}{2} \cdot 8\right)^2\right) = 272 + 16\left(\frac{1}{2} \cdot 2\right)^2 - 9\left(\frac{1}{2} \cdot 8\right)^2$$

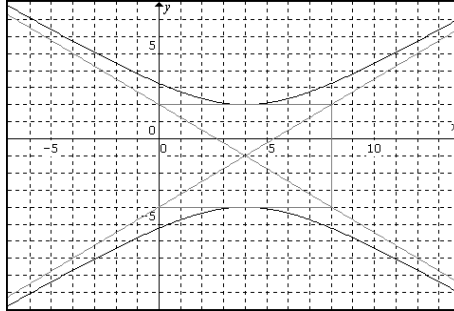
$$16(y+1)^2 - 9(x-4)^2 = 272 + 16 - 144$$

$$16(y+1)^2 - 9(x-4)^2 = 144$$

$$\frac{16(y+1)^2}{144} - \frac{9(x-4)^2}{144} = 1$$

$$\frac{(y+1)^2}{9} - \frac{(x-4)^2}{16} = 1$$

Now we can determine our center. Remember, the  $x$  goes with  $x$  and the  $y$  goes with  $y$ . So our center is  $(4, -1)$ . Also we can determine  $a$  and  $b$ . Recall that  $a$  is always under the  $x$  part and  $b$  is always under the  $y$  part. Thus,  $a = 4$  and  $b = 3$ . We can use these with the fact that the  $y$  part is positive so the axis is vertical to graph. We get



So the vertices are  $(4, 2)$  and  $(4, -4)$ .

Now all we need is our intercepts. Looking at the graph we should have no  $x$ -intercepts and two  $y$ -intercepts. Lets find them as usual.

$x$ -intercepts:

$$\begin{aligned} \frac{(0+1)^2}{9} - \frac{(x-4)^2}{16} &= 1 \\ 16 - 9(x-4)^2 &= 144 \\ -9(x-4)^2 &= 128 \\ (x-4)^2 &= -\frac{128}{9} \\ x &= 4 \pm \sqrt{-\frac{128}{9}} \end{aligned}$$

$y$ -intercepts:

$$\begin{aligned} \frac{(y+1)^2}{9} - \frac{(0-4)^2}{16} &= 1 \\ 16(y+1)^2 - 144 &= 144 \\ 16(y+1)^2 &= 288 \\ (y+1)^2 &= 18 \\ y &= -1 \pm 3\sqrt{2} \\ y &\approx -5.2, 3.2 \end{aligned}$$

Since the  $x$ -intercepts are complex numbers we have no  $x$ -intercepts as we thought. We can clearly see the  $y$ -intercepts are consistent with what is graphed above.

Now that we have a familiarity with all of our conic sections we need to be able to distinguish between all the various types.

From the last section we were able to distinguish between the parabola, circle and ellipse. The hyperbola is always easy to spot. It is the only conic section that has both variable squared and one of them has a negative coefficient. So we have the following general guidelines.

### Distinguishing Between Various Conic Sections

If only one of the two variables is squared, the conic is a parabola.

If both variables are squared, then we have the following:

- If the square terms are the same sign (both positive or both negative) and have the same coefficients, the conic is a circle.
- If the square terms are the same sign (both positive or both negative) and have different coefficients, the conic is an ellipse.
- If the square terms are different signs (one is positive and the other is negative), the conic is a hyperbola.

Example 3:

Identify the type of conic the equation represents, and then graph accordingly.

a.  $4x + y^2 - 2y - 33 = 0$                       b.  $2x^2 + 8y^2 - 4x + 16y - 22 = 0$

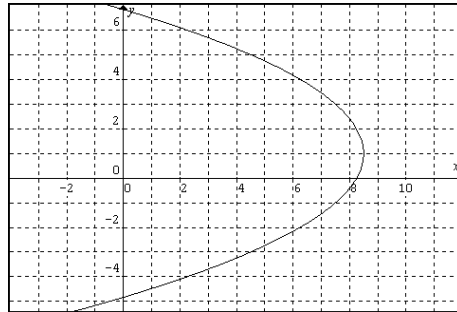
Solution:

- a. Since only one of the variables is squared, we know that the equation represents a parabola. Now we simply need to get it into standard form and graph like we learned earlier in this chapter. We proceed as follows

$$\begin{aligned}4x + y^2 - 2y - 33 &= 0 \\4x &= -y^2 + 2y + 33 \\4x &= -(y^2 - 2y) + 33 \\x &= -\frac{1}{4}(y^2 - 2y) + \frac{33}{4} \\x &= -\frac{1}{4}\left(y^2 - 2y + \left(\frac{1}{2} \cdot 2\right)^2\right) + \frac{33}{4} - \left(-\frac{1}{4}\left(\frac{1}{2} \cdot 2\right)^2\right) \\x &= -\frac{1}{4}(y - 1)^2 + 8\frac{1}{2}\end{aligned}$$

So, our vertex is at  $(8\frac{1}{2}, 1)$  (always remember that the number that is with y is the y value of the vertex or center and likewise for x).

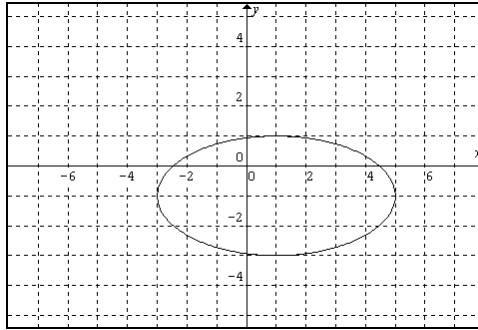
Now we can simply plot the vertex and use the fact that it opens left to finish the graph.



- b. Since both variables are squared we either have a circle, ellipse or hyperbola. Since the coefficients of the squared terms are both positive (the same sign) and different values, we must be dealing with an ellipse. Thus, we will put it in standard form to determine the center. We get

$$\begin{aligned}2x^2 + 8y^2 - 4x + 16y - 22 &= 0 \\2x^2 - 4x + 8y^2 + 16y &= 22 \\2(x^2 - 2x) + 8(y^2 + 2y) &= 22 \\2\left(x^2 - 2x + \left(\frac{1}{2} \cdot 2\right)^2\right) + 8\left(y^2 + 2y + \left(\frac{1}{2} \cdot 2\right)^2\right) &= 22 + 2\left(\frac{1}{2} \cdot 2\right)^2 + 8\left(\frac{1}{2} \cdot 2\right)^2 \\2(x - 1)^2 + 8(y + 1)^2 &= 22 + 2 + 8 \\2(x - 1)^2 + 8(y + 1)^2 &= 32 \\ \frac{(x - 1)^2}{16} + \frac{(y + 1)^2}{4} &= 1\end{aligned}$$

So, the center is  $(1, -1)$ ,  $a = 4$  and  $b = 2$ . So we graph as before to get



## 12.5 Exercises

Graph the following. Find the center, vertices and x- and y-intercepts.

1.  $\frac{x^2}{9} - \frac{y^2}{4} = 1$
2.  $\frac{x^2}{4} - \frac{y^2}{4} = 1$
3.  $16y^2 - x^2 = 16$
4.  $y^2 - x^2 = 1$
5.  $\frac{(y-1)^2}{16} - \frac{(x-2)^2}{4} = 1$
6.  $\frac{(x-3)^2}{4} - \frac{(y-2)^2}{9} = 1$
7.  $\frac{(x+1)^2}{4} - \frac{(y-3)^2}{16} = 1$
8.  $\frac{(y+2)^2}{16} - \frac{(x+1)^2}{9} = 1$
9.  $(x-1)^2 - (y-1)^2 = 1$
10.  $(y+2)^2 - \frac{(x-1)^2}{4} = 1$
11.  $9(x-3)^2 - 4(y+3)^2 = 36$
12.  $4(x+2)^2 - (y+2)^2 = 16$
13.  $(y-2)^2 = 4(x-1)^2 + 4$
14.  $36(y+2)^2 = 16(x-3)^2 + 144$
15.  $(y+2)^2 = 4x^2 + 36$
16.  $x^2 = 9(y-2)^2 + 36$
17.  $4x^2 - 9y^2 + 16x + 18y - 29 = 0$
18.  $4x^2 - y^2 + 32x + 6y + 51 = 0$
19.  $y^2 - 9x^2 + 8y + 54x - 74 = 0$
20.  $4y^2 - 9x^2 - 8y - 36x - 68 = 0$
21.  $y^2 - 4x^2 + 24x + 2y - 51 = 0$
22.  $4y^2 - x^2 - 8y + 2x - 13 = 0$
23.  $9x^2 - y^2 - 54x - 2y + 44 = 0$
24.  $4x^2 - y^2 + 16x + 2y - 21 = 0$
25.  $x^2 - y^2 - 8x + 4y + 8 = 0$
26.  $x^2 - y^2 - 6x - 4y - 4 = 0$
27.  $y^2 - x^2 - 4x - 5 = 0$
28.  $y^2 - x^2 - 2y = 0$
29.  $16y^2 - 25x^2 - 32y - 50x - 409 = 0$
30.  $25y^2 - 16x^2 + 50y + 64x - 439 = 0$
31.  $25x^2 - 4y^2 + 50x - 24y - 111 = 0$
32.  $9x^2 - 25y^2 - 36x + 150y - 414 = 0$
33.  $4x^2 + 36y = 9y^2 + 8x + 68$
34.  $9x^2 + 36x = 16y^2 + 128x + 364$
35.  $9y^2 + 36y - 36 = 4x^2 + 24x$
36.  $y^2 + 6 = x^2 + 8y + 2x$
37.  $y^2 = 9x^2 + 18x + 6y + 9$
38.  $4y^2 + 4x = x^2 + 8y + 2x$
39.  $x^2 - y^2 = 2x - 2y + 1$
40.  $x^2 - y^2 = 4y - 4x + 1$

Identify the type of conic the equation represents, and then graph accordingly.

41.  $x = y^2 - 2y + 3$

42.  $25y^2 + 4x^2 = 100$

43.  $x^2 + y^2 + 6x - 8y + 21 = 0$

44.  $4(x-1)^2 = (y+1)^2 + 4$

45.  $x^2 - y^2 + 4x + 5 = 0$

46.  $\frac{(x-3)^2}{9} + \frac{(y+4)^2}{25} = 1$

47.  $x^2 + 9y^2 + 4x - 18y - 72 = 0$

48.  $x^2 + y^2 - 4x + 6y - 3 = 0$

49.  $4x^2 - y^2 - 8x - 14y - 49 = 0$

50.  $x = -2y^2 + 4y - 6$

51.  $9x^2 + 4y^2 + 54x - 32y + 109 = 0$

52.  $9x^2 - 4y^2 + 18x - 8y + 41 = 0$

53.  $x^2 + 2y - 12 = 0$

54.  $x^2 - 8x + y^2 + 10y = 0$

55.  $y^2 - 25x^2 + 8y - 9 = 0$

56.  $x^2 + 6x - 12y + 33 = 0$

Find the equation of the hyperbola that satisfies the given conditions.

57. center  $(-1, 2)$ ,  $a = 2$ ,  $b = 3$ , horizontal axis

58. center  $(3, -1)$ ,  $a = 4$ ,  $b = 1$ , vertical axis

59. vertices  $(-1, 3)$ ,  $(-1, 7)$  and  $(0, 0)$  is on the graph

60. vertices  $(2, 4)$ ,  $(-2, 4)$  and  $(6, 0)$  is on the graph

61.  $a = 1$ ,  $b = 1$ , vertical axis,  $(-1, 3)$  is a vertex

62.  $a = 4$ ,  $b = 3$ , horizontal axis,  $(2, 2)$  is a vertex