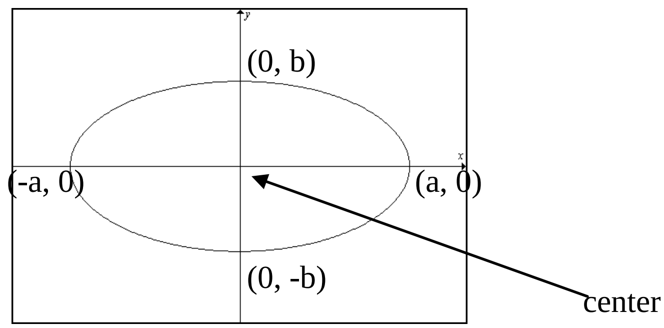
12.4 The Ellipse

The next one of our conic sections we would like to discuss is the ellipse. We will start by looking at the ellipse centered at the origin and then move it away from the origin.

Standard Form of an Ellipse Centered at (0, 0)
The equation of an ellipse with center (0, 0) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The x-intercepts are (a, 0) and (-a, 0), and the y-intercepts are (0, b) and (0, -b).

This definition gives us the following graph



So in order to graph an ellipse we simply need to go right and left, *a* units from the center and up and down, *b* units from the center. This is easy to remember since the *a* value is beneath x in the equation and the *b* value is beneath the y in the equation and x's go left and right and y's go up and down.

An ellipse can either be horizontal as the graph above, or can be vertical. The way we distinguish between these is by the values of *a* and *b*. Which ever is larger will determine the placement of the ellipse as we will see in the following example.

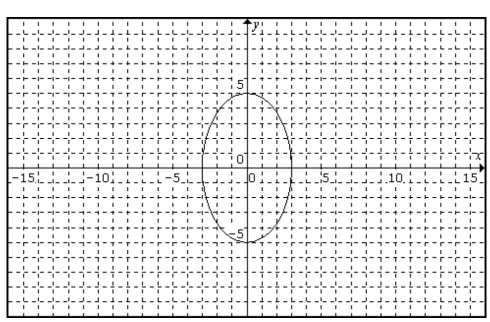
<u>Example 1</u>:

Graph the following.

a. $\frac{x^2}{9}+\frac{y^2}{25}=1$ 	b. $4x^2+9y^2-36=0$

Solution:

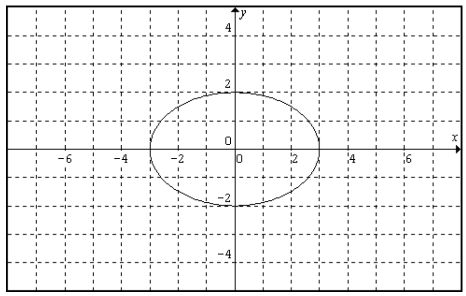
a. First we notice that the equation is already in standard form. Clearly, *a* = 3 and *b* = 5. So since the 9 is under the x, we go 3 units in the x directions both ways (right and left). Similarly, 5 units in y directions both ways (up and down). Then simply connect the points to get the following



b. This time the equation is not in standard form. So we must change it into standard form. We proceed as follows

$$4x^2 + 9y^2 - 36 = 0$$
$$4x^2 + 9y^2 = 36$$
$$\frac{4x^2}{36} + \frac{9y^2}{36} = 1$$
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

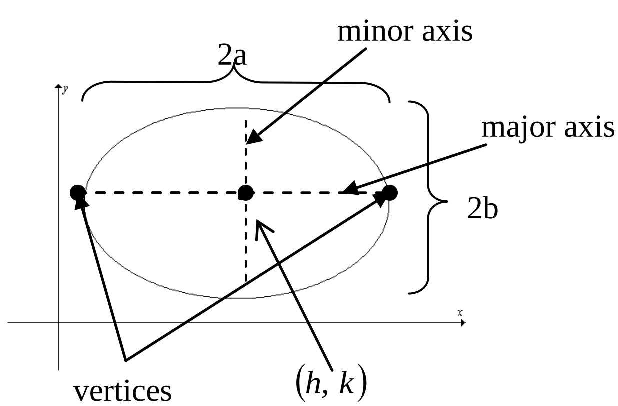
So now we see that *a* = 3 and *b* = 2. That is we go 3 units right and left and 2 units up and down. This gives



As we know, not everything is life is centered at the origin. So therefore we need to discuss the ellipse that is centered at anywhere on the xy-plane.

Standard Form of an Ellipse
The equation of the ellipse with center (h, k) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The longer axis is called the <u>major axis</u>, the shorter axis is called the <u>minor axis</u>. The endpoints of the major axis are called the <u>vertices</u>.

This definition gives us the following graph



The graph of an ellipse where the major axis is vertical is similar. The value of *b* would simply be larger then the value of *a*.

Similar to the case where the center was at the origin, to graph these we simply start at the center and go *a* units in the x directions (left and right) and *b* units in the y directions (up and down). The vertices are always on the endpoints of the major axis.

Example 2:

Graph the following. Find the center, vertices and x-and y-intercepts.

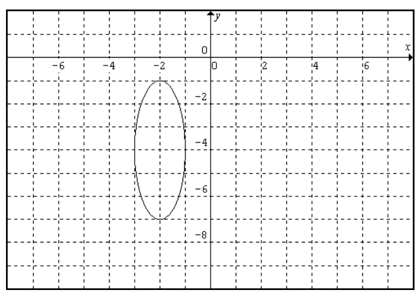
a. $(x+2)^2 + \frac{(y+4)^2}{9} = 1$ b. $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

Solution:

a. First we see that the equations appears to be in standard form, however, there is no denominator for the front part. But recall that anything is the same as itself over 1. So we can look at the equation as

$$\frac{(x+2)^2}{1} + \frac{(y+4)^2}{9} = 1$$

So now we can determine the center and our *a* and *b* values. Similar to the circle we carefully extract the center and get $(-2, -4)$. Also, we can see that we have *a* = 1 and *b* = 3. Now before we determine the vertices and intercepts its easiest to graph at this point and then we can simply read the vertices off the graph. So we start at the center and go 1 to the right and left (since that is the x directions) and 3 up and down (since that was the y directions). Its always easy to remember that you just go in the direction of the variable that had the value underneath it. So we get



Since the major axis is clearly vertical here, and we know the vertices are always at the endpoints of the major axis, we can just read the vertices off the graph. We get $(-2, -1)$ and $(-2, -7)$.

Now all we need is the intercepts. However, we can see that the graph does not hit either axis and therefore should not have any intercepts. We can verify this by calculating them as we usually do. That is, for x-intercepts, let y = 0 and for y-intercepts, let x = 0.

x-intercepts:

$$(x+2)^2 + \frac{(0+4)^2}{9} = 1$$
$$9(x+2)^2 + 16 = 9$$
$$9(x+2)^2 = -7$$
$$x = -2 \pm \sqrt{-\tfrac{7}{9}}$$

y-intercepts:

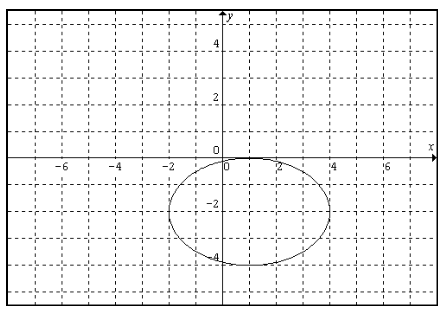
$$(0+2)^2 + \frac{(y+4)^2}{9} = 1$$
$$36 + (y+4)^2 = 9$$
$$(y+4)^2 = -27$$
$$y = -4 \pm \sqrt{-27}$$

Since we have complex solutions one each of these we clearly have no x- or y-intercepts as we had seen from the graph.

b. This time we have an equation that is not is standard form. So we must start by getting it into standard form by completing the square as we did in the last section only this time we need to be extra careful with the coefficients in front of the squared variables. So we factor them out first and when we add the completing the square piece to both sides we need to include that coefficient as well. We proceed as follows

$$4x^2 + 9y^2 - 8x + 36y + 4 = 0$$
$$4x^2 - 8x + 9y^2 + 36y = -4$$
$$4\left(x^2 - 2x \quad\right) + 9\left(y^2 + 4y \quad\right) = -4$$
$$4\left(x^2 - 2x + \left(\tfrac{1}{2}\cdot 2\right)^2\right) + 9\left(y^2 + 4y + \left(\tfrac{1}{2}\cdot 4\right)^2\right) = -4 + 4\left(\tfrac{1}{2}\cdot 2\right)^2 + 9\left(\tfrac{1}{2}\cdot 4\right)^2$$
$$4(x-1)^2 + 9(y+2)^2 = -4 + 4 + 36$$
$$4(x-1)^2 + 9(y+2)^2 = 36$$
$$\frac{4(x-1)^2}{36} + \frac{9(y+2)^2}{36} = 1$$
$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Now we can determine our center. It is $(1, -2)$. Also we can graph using *a* = 3 and *b* = 2. This gives us



So our vertices are always at the ends of the major axis (which is horizontal this time). So the vertices are $(-2, -2)$ and $(4, -2)$.

Now all we need is our intercepts. Looking at the graph we should have one x-intercept and two y-intercepts. Lets find them as usual.

x-intercepts:

$$\frac{(x-1)^2}{9} + \frac{(0+2)^2}{4} = 1$$
$$4(x-1)^2 + 36 = 36$$
$$4(x-1)^2 = 0$$
$$(x-1)^2 = 0$$
$$x = 1 \pm \sqrt{0}$$
$$x = 1$$

y-intercepts:

$$\frac{(0-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$
$$4 + 9(y+2)^2 = 36$$
$$9(y+2)^2 = 32$$
$$(y+2)^2 = \tfrac{32}{9}$$
$$y = -2 \pm \tfrac{4\sqrt{2}}{3}$$
$$y \approx -3.9,\ 0.1$$

We can clearly see these are the values that are graphed above.

As we stated in the last section, one of the main things that we want to do in this chapter is to be able to distinguish between the various conics from the very beginning of the problem. So now we have three different conics: the parabola, circle and ellipse. The parabola will always be easy tell from all the other conics, since the parabola will be the only one that only has one variable that is squared. However, to tell the difference between the circle and ellipse is slightly more challenging. The only difference between a circle and ellipse is that the circle has the same coefficients for x^2 and y^2 and the ellipse has different coefficients.

So, this gives us the following
Only one of the two variables is squared- the conic is a parabola
Both variables are squared and the coefficients of them is the same- the conic is a circle
Both variables are squared and the coefficients of them are the different- the conic is an ellipse.

12.4 Exercises

Graph the following. Find the center, vertices and x-and y-intercepts.

1. $\frac{x^2}{4}+\frac{y^2}{9}=1$
2. $\frac{x^2}{9}+\frac{y^2}{4}=1$
3. $x^2+36y^2=36$
4. $4x^2+y^2-36=0$
5. $\frac{(x-1)^2}{16}+\frac{(y-2)^2}{4}=1$
6. $\frac{(x-3)^2}{4}+\frac{(y-2)^2}{9}=1$
7. $\frac{(x+1)^2}{4}+\frac{(y-3)^2}{16}=1$
8. $\frac{(x+2)^2}{9}+\frac{(y+1)^2}{16}=1$
9. $\frac{(x+2)^2}{9}+(y-5)^2=1$
10. $(x-2)^2+\frac{(y-2)^2}{4}=1$
11. $9(x-3)^2+4(y+3)^2=36$
12. $4(x+2)^2+(y+2)^2=16$
13. $(x-2)^2+4(y-2)^2=4$
14. $36(x+2)^2+16(y-3)^2-144=0$
15. $(x+2)^2+4y^2-36=0$
16. $x^2+9(y-2)^2-36=0$
17. $9x^2+4y^2-18x-16y-11=0$
18. $4x^2+9y^2-16x-54y+61=0$
19. $x^2+4y^2+2x-24y+21=0$
20. $4x^2+y^2+8x-4y-8=0$
21. $x^2+9y^2+8x+36y+43=0$
22. $9x^2+4y^2-36x+24y+36=0$
23. $16x^2+y^2-32x-2y+1=0$
24. $x^2+4y^2-6x+24y+41=0$
25. $9x^2+4y^2-8y-32=0$
26. $9x^2+4y^2-18x-27=0$
27. $x^2+4y^2-4x=0$
28. $9x^2+y^2-6y=0$
29. $x^2+4y^2+6x+16y-11=0$
30. $x^2+9y^2-4x+18y-23=0$
31. $4x^2+y^2-24x+10y+45=0$
32. $4x^2+y^2+32x-6y+57=0$
33. $9x^2+4y^2=90x+24y-225$
34. $9x^2+4y^2+40y=72x-208$
35. $x^2+16+4y^2=8x-8y$
36. $9x^2+y^2+76=54x+4y$
37. $9x^2+4y^2+16=36x+16y$
38. $4x^2+9y^2+16x+36y=-16$
39. $16x^2+9y^2+64x=108y-244$
40. $9x^2+16y^2+90x=128y-337$

Identify the type of conic the equation represents, and then graph accordingly.

41. $x^2+y^2-4x+6y+9=0$
42. $x^2+y^2-4x-2y-20=0$

43. $2y^2 - 4y + x - 7 = 0$

44. $x^2 + 36y^2 - 72y = 0$

45. $y^2 - y - x = 0$

46. $x^2 + y^2 + 4x - 2y - 11 = 0$

47. $4x^2 + 9y^2 + 8x - 32 = 0$

48. $y^2 - 2y + x - 3 = 0$

49. $x^2 + y^2 + 2x - 8y + 8 = 0$

50. $9x^2 + 4y^2 + 18x - 32y + 37 = 0$

51. $x^2 + 9y^2 + 2x - 18y - 26 = 0$

52. $3y^2 - x + 6y - 1 = 0$

Find the equation of the ellipse that satisfies the given conditions.

53. center $(2, -3)$, $a = 2$, $b = 3$

54. center $(-1, 2)$, $a = 3$, $b = 4$

55. vertices $(5, 4)$, $(-1, 4)$; $(0, 5)$ is on the graph

56. vertices $(1, -2)$, $(1, 2)$; $(0, 0)$ is on the graph

57. vertices $(3, 2)$, $(-1, 2)$; minor axis is length 2

58. vertices $(-1, 7)$, $(-1, 1)$; minor axis is length 4

The formula for the area of an ellipse is $A = \pi ab$. Find the area of the following ellipses.

59. $\frac{(x-2)^2}{16} + \frac{(y-2)^2}{4} = 1$

60. $\frac{(x+5)^2}{9} + (y-4)^2 = 1$

61. $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

62. $9x^2 + 4y^2 - 18x - 16y - 11 = 0$

63. $x^2 + 4y^2 + 2x - 24y + 21 = 0$

64. $x^2 + 9y^2 + 8x + 36y + 43 = 0$