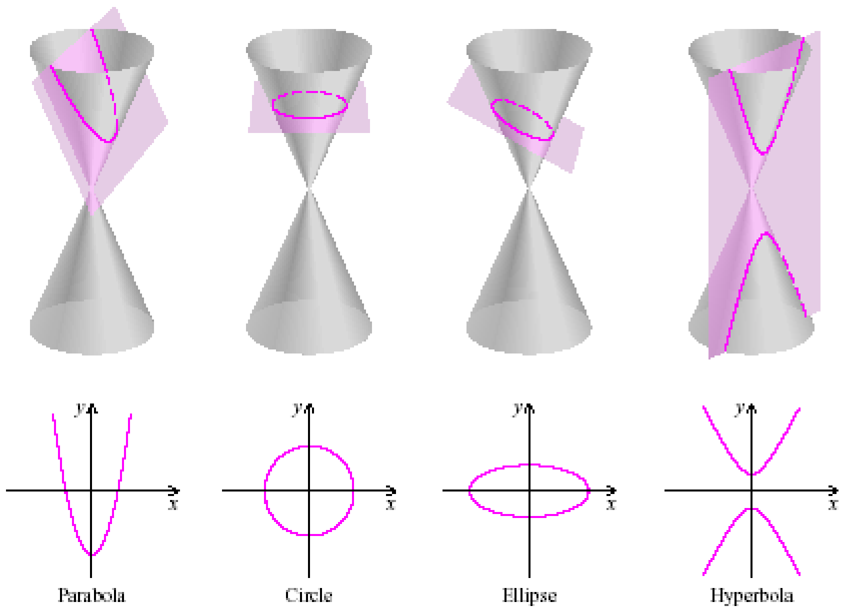
12.2 The Parabola

The main topic of this chapter is something we call conic sections.

Definition: Conic Sections
The conic sections are all the curves that can be made from slicing two cones stacked tip to tip with a plane.

Some types of conic sections (or conics) we have already seen. For example, if we slice the cones through the tips we get a single point. If we slice the cones in such a way as we simply touch one side of one cone and the opposite side of the other, we get a line. Lastly, if we slice them vertically though the tips, we get two intersecting lines. These types of conics we have already dealt with. We are more interested four conic sections demonstrated below.



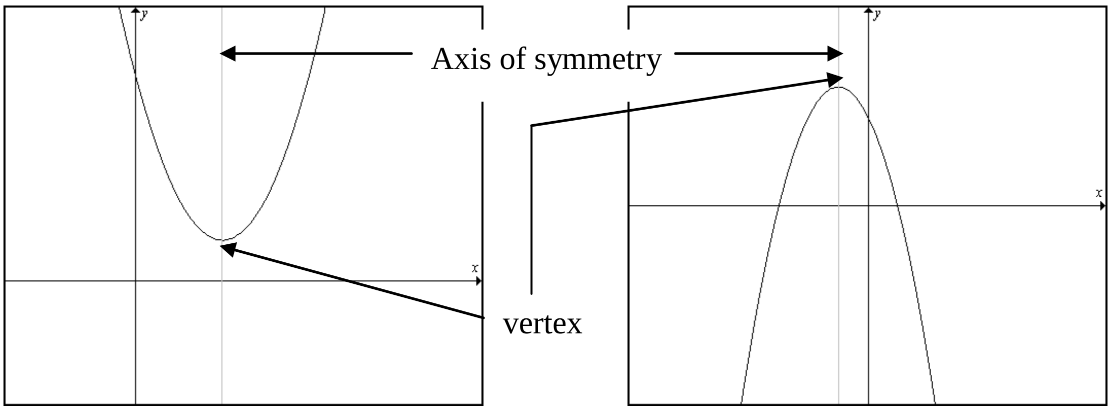
We will start with the parabola since we have already had some contact with parabolas in chapter 10.

Standard Form of a Parabola
The equation of a parabola with vertex at (h,k) is $y=a(x-h)^2+k$ for a vertical axis of symmetry of $x=h$ or, $x=a(y-k)^2+h$ for a horizontal axis of symmetry of $y=k$.

This gives us the following four possibilities for parabolas, the first two we have seen before.

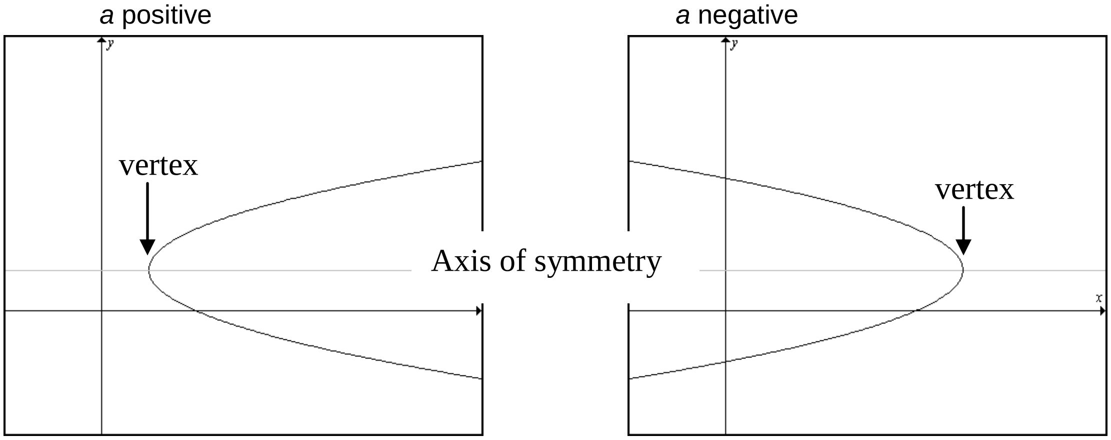
$$y = a(x-h)^2 + k$$

a positive *a* negative



$$x = a(y-k)^2 + h$$

a positive *a* negative



We can see that graphing these parabolas will be very similar to how we graphed them in chapter 10. Its just that now its possible for the graph to be sideways.

<u>Example 1</u>:

Graph the following. State the vertex and all intercepts.

a. $y = 2x^2 + 4x - 1$ b. $x = y^2 + 6y + 5$

Solution:

a. First we need to get the parabola into standard form so we can find the vertex. We do this by completing the square (recall there is an alternate method of using x = -b/2a, however, for the purpose of this chapter its best to do use the completing the square method. This way, all conics can be treated similarly). We proceed as follows

$$
\begin{aligned}
y &= 2x^2 + 4x - 1 \\
&= 2\left(x^2 + 2x \qquad\right) - 1 \\
&= 2\left(x^2 + 2x + \left(\tfrac{1}{2}\cdot 2\right)^2\right) - 1 - 2\left(\tfrac{1}{2}\cdot 2\right)^2 \\
&= 2(x+1)^2 - 3
\end{aligned}
$$

So our vertex is at $(-1, -3)$

Next we need our intercepts. Recall, to find the x-intercepts, we set the bottom equation from above to equal 0 and to find the y-intercepts we set x in the top equation to zero. This gives us

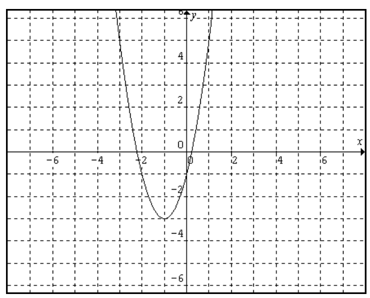
x-intercepts:

$$
\begin{aligned}
2(x+1)^2 - 3 &= 0 \\
(x+1)^2 &= \tfrac{3}{2} \\
x &= -1 \pm \sqrt{\tfrac{3}{2}} \\
x &\approx -2.2,\ 0.2
\end{aligned}
$$

y-intercept:

$$
\begin{aligned}
y &= 2(0)^2 + 4(0) - 1 \\
&= -1
\end{aligned}
$$

Now we simply plot all of our values and use the fact that the graph opens up (since a = 2 which means a > 0) to complete the graph.



b. Similarly we need to start by getting the equation into standard form. So we complete the square on the y terms this time to get the following

$$
\begin{aligned}
x &= y^2 + 6y + 5 \\
&= \left(y^2 + 6y + \left(\tfrac{1}{2}\cdot 6\right)^2\right) + 5 - \left(\tfrac{1}{2}\cdot 6\right)^2 \\
&= (y+3)^2 - 4
\end{aligned}
$$

Notice, though, the order here for the vertex is reversed. This time we have the vertex is $(-4, -3)$. We can always remember this by simply remembering that the value inside the parenthesis always goes with the variable inside the parenthesis and that its sign is always opposite. The value on the outside always goes with the other variable. So here, the 3 goes with the y as a –3 and so thus –4 goes with the x.

Now again we need the intercepts this time the way we do it is again switched. This time to find the y-intercepts, we set the bottom equation from above to equal 0 and to find the x-intercepts we set y in the top equation to zero.

This gives us

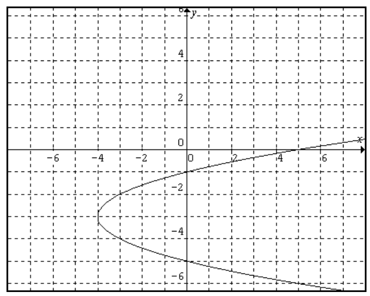
x-intercept:

$$x = (0)^2 + 6(0) + 5$$
$$= 5$$

y-intercepts:

$$(y+3)^2 - 4 = 0$$
$$(y+3)^2 = 4$$
$$y = -3 \pm 2$$
$$y = -5,\ -1$$

Now we simply plot all of our values and use the fact that the graph opens right (since a = 1 which means a > 0) to complete the graph.



Example 2:

Graph the following. State the vertex and all intercepts.

a. $y = -\frac{1}{2}x^2 + 2x + 6$ b. $x = -y^2 + 2y - 3$

Solution:

a. Proceeding like in example 1 we complete the square to find the vertex. We get

$$y = -\tfrac{1}{2}x^2 + 2x + 6$$
$$= -\tfrac{1}{2}\left(x^2 - 4x \qquad\right) + 6$$
$$= -\tfrac{1}{2}\left(x^2 - 4x + \left(\tfrac{1}{2}\cdot 4\right)^2\right) + 6 - \left(-\tfrac{1}{2}\left(\tfrac{1}{2}\cdot 4\right)^2\right)$$
$$= -\tfrac{1}{2}(x-2)^2 + 8$$

So the vertex is at $(2, 8)$. Next we find the intercepts.

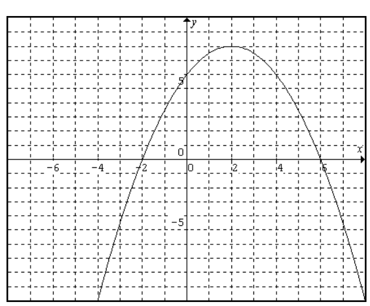
x-intercepts:

$$-\tfrac{1}{2}(x-2)^2 + 8 = 0$$
$$(x-2)^2 = 16$$
$$x = 2 \pm 4$$
$$x = -2,\ 6$$

y-intercept:

$$y = -\tfrac{1}{2}(0)^2 + 2(0) + 6$$
$$= 6$$

Graphing we get



b. Similarly we find the vertex.

$$\begin{aligned} x &= -y^2 + 2y - 3 \\ &= -\left(y^2 - 2y \qquad\right) - 3 \\ &= -\left(y^2 - 2y + \left(\tfrac{1}{2}\cdot 2\right)^2\right) - 3 - \left(-\left(\tfrac{1}{2}\cdot 2\right)^2\right) \\ &= -\left(y-1\right)^2 - 2 \end{aligned}$$

So the vertex is at $(-2, 1)$. Next we find the intercepts.

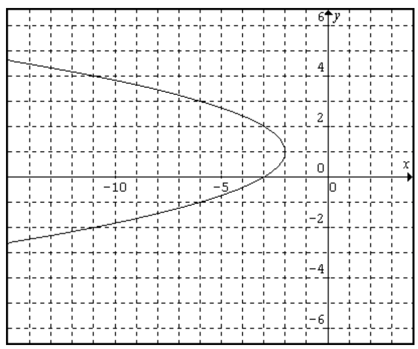
x-intercept:

$$\begin{aligned} x &= -(0)^2 + 2(0) - 3 \\ &= -3 \end{aligned}$$

y-intercepts:

$$\begin{aligned} -\left(y-1\right)^2 - 2 &= 0 \\ \left(y-1\right)^2 &= -2 \\ y &= 1 \pm i\sqrt{2} \end{aligned}$$

Since the y intercepts are complex numbers there must be no y-intercepts. So putting it all together and using the fact that the graph opens left (since a < 0) we get



12.2 Exercises

Graph the following. State the vertex and all intercepts.

1. $y = (x-5)^2 - 1$
2. $y = (x+3)^2 - 9$
3. $x = (y+1)^2 - 4$
4. $x = (y-3)^2 - 16$
5. $x = (y+4)^2 - 3$
6. $x = (y-2)^2 - 5$
7. $y = (x+2)^2 - 2$
8. $y = (x-4)^2 - 3$
9. $y = -2(x-1)^2 - 3$
10. $y = -3(x+1)^2 + 1$
11. $x = -(y-2)^2 + 4$
12. $x = -2(y-1)^2 + 7$
13. $x = 3(y+3)^2 + 2$
14. $x = -2(y-2)^2 - 2$
15. $y = -\frac{3}{2}x^2 + 3$
16. $y = \frac{1}{2}(x+2)^2$
17. $y = x^2 + 4x + 5$
18. $y = x^2 + 2x - 5$
19. $x = y^2 - 6y + 13$
20. $x = y^2 - 4y + 5$
21. $x = y^2 + 8y + 20$
22. $x = y^2 - 10y + 21$
23. $y = 2x^2 + 16x + 23$
24. $y = 2x^2 - 16x + 25$
25. $x = -y^2 - 2y + 7$
26. $x = -y^2 + 2y + 5$
27. $x = 3y^2 - 24y + 50$
28. $x = 4y^2 + 8y - 3$
29. $y = x^2 + 7x$
30. $y = x^2 - 5x$
31. $y = -2x^2 - 4x - 6$
32. $y = -3x^2 + 6x + 2$
33. $x = 2y^2 + 5y - 3$
34. $x = 4y^2 + 2y - 1$
35. $x = -3y^2 + 5y - 2$
36. $x = -3y^2 - 7y + 2$
37. $y = \frac{1}{2}x^2 + 4x + \frac{19}{3}$
38. $y = \frac{1}{2}x^2 - 3x + 2$
39. $x = -\frac{1}{3}y^2 - 5y + 3$

40. $x=-\frac{2}{3}y^2+4y-1$

41. $y=\frac{x^2}{2}+x-\frac{3}{2}$

42. $x=\frac{y^2}{2}+3y-\frac{4}{3}$

43. $2x^2-12x-y+18=0$

44. $2y^2-x+1=0$

45. $x^2-y+7=3x^2+2y$

46. $x-y^2=2x+y-3$

47. $y+(x-2)^2=-2y+2x^2$

48. $(y+1)^2=x+y-1$

Find the equation of the parabola in standard form which satisfies the following.

49. vertex $(-1, 3)$, $a=3$, vertical axis of symmetry
50. vertex $(2, -4)$, $a=-1$, vertical axis of symmetry
51. vertex $(5, -2)$, runs through the point $(1, 1)$, horizontal axis of symmetry
52. vertex $(-3, 5)$, runs through the point $(-2, 6)$, horizontal axis of symmetry
53. vertex is at the midpoint of $(3, 2)$ and $(9, 8)$, runs through the point $(9, 8)$, horizontal axis of symmetry
54. vertex is at the midpoint of $(-4, 0)$ and $(5, -3)$, runs through the point $(-4, 0)$, horizontal axis of symmetry
55. axis of symmetry $x=3$, vertex is at the intersection of the axis of symmetry and the line $2x+3y=6$, runs through the point $(-2, 6)$
56. axis of symmetry $y=-1$, vertex is at the intersection of the axis of symmetry and the line $x-3y=5$, runs through the point $(-1, 3)$