# 11.6 Applications of Exponentials and Logarithms

In this section we want to apply all that we have learned about logarithms and exponentials to real situations. In order to do this we first need some models.

**Compound Interest**

The amount $A$ after time $t$, with interest rate $r$, and principle $P$ is given by the following:

For $n$ compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

For continuous compounding: $A = Pe^{rt}$

**Richter Scale Intensity Model**

$M = \log \frac{I}{I_0}$, where $M$ is the magnitude of the earthquake, $I$ is the intensity measured by the amplitude of the seismograph 100 km from the epicenter (expressed in cm), and $I_0$ is a constant.

**pH Model**

$pH = -\log\left(H^+\right)$, where $H^+$ is the hydrogen ion concentration (in moles/liter).

**Sound Intensity Model**

$L = 10 \cdot \log \frac{I}{I_0}$ where *L* is the loudness (in decibels), $I$ is the intensity of the sound (in watts/m$^2$), and $I_0 = 10^{-12}$

**Newtons Law of Cooling**

$T = T_s + D_0 e^{-kt}$ where *T* is the temperature after time *t*, $T_s$ is the temperature of the surrounding environment, $D_0$ is the initial temperature and *k* is a constant.

The final model is the most important model.

**Exponential Growth and Decay Model**

The model for growth and decay is given by

$$y = Ce^{kt},$$

where $t$ is time, $C$ is the original amount, $y$ is the amount after time $t$ and $k$ is a constant determined by the rate of growth.

This model is important because it shows up quite often in population growth.

There are many other models besides these. However, these have the most useful applications and are therefore worthwhile highlighting. Other applications range from spread of a disease to statistics.

We will concentrate mostly on the models given.

<u>Example 1</u>:

An investment of $50,000 is made in an account that compounds interest quarterly. After 6 years, the balance in the account is $62,748.37. What is the annual interest rate for this account?

Solution:

We need the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$. Let us label what we have.

$P = \$50{,}000$

$A = \$62{,}748.37$

$t = 6\ years$

$n = 4$

So into the formula we get

$$62748.37 = 50000\left(1+\frac{r}{4}\right)^{4\cdot 6}$$

Now just use previous techniques to solve for r.

$$\frac{62748.37}{50000} = \left(1+\frac{r}{4}\right)^{4\cdot 6}$$

$$1.2549674 = \left(1+\frac{r}{4}\right)^{24}$$

$$(1.2549674)^{1/24} = 1+\frac{r}{4}$$

$$1.0095 \approx 1+\frac{r}{4}$$

$$0.038 \approx r$$

Thus, the interest rate is about 3.8%.

<u>Example 2</u>:

Fred invested $30,000 into a mutual fund that compounds continuously at a rate of 6.2%. How long will it take before Fred has $48,950?

Solution:

We need the formula $A = Pe^{rt}$. We are looking for $t$.

$A = \$48{,}950$

$P = \$30{,}000$

$r = 0.062$

$$48950 = 30000e^{0.062t}$$
$$\frac{48950}{30000} = e^{0.062t}$$
$$\ln\frac{48950}{30000} = 0.062t$$
$$\frac{\ln\frac{48950}{30000}}{0.062} = t$$
$$7.8968 \approx t$$

So, Fred must wait about 7 years, 11 months.

<u>Example 3</u>:

In October 1989, San Francisco, CA had an earthquake that measured 7.1 on the Richer scale. In January 1994, an earthquake in Northridge, CA measured 6.6 on the Richer scale. Compare the intensities of the two earthquakes.

Solution:

Notice we have two different situations here. First we have the San Francisco quake and then we also have the Northridge quake. So in order to compare these earthquakes lets first put our values into the formula for earthquake intensity. Since we have two different earthquakes lets label the intensity of the San Francisco earthquake as $I_{SF}$ and likewise Northridge's intensity as $I_N$. So we have the following

$$7.1 = \log\frac{I_{SF}}{I_0} \qquad \text{and} \qquad 6.6 = \log\frac{I_N}{I_0}$$

So we have two different equations with a total of three variables. Lets get rid of one of the variables, the $I_0$. Heres how we do that. Lets first expand each of the equations. This gives us

$$7.1 = \log I_{SF} - \log I_0 \qquad \text{and} \qquad 6.6 = \log I_N - \log I_0$$

Now we subtract the two equations to eliminate the $I_0$. We have

$$\begin{array}{r} 7.1 = \log I_{SF} - \log I_0 \\ \underline{-6.6 = -\log I_N + \log I_0} \\ 0.5 = \log I_{SF} - \log I_N \end{array}$$

Now if we recondense the resulting equation we have

$$0.5 = \log\frac{I_{SF}}{I_N}$$

So we can solve for the ratio $I_{SF} / I_N$ which will give us a nice comparison of the two earthquakes. So we get

$$
\begin{aligned}
0.5 &= \log \frac{I_{SF}}{I_N} \\
10^{0.5} &= \frac{I_{SF}}{I_N} \\
10^{0.5} I_N &= I_{SF} \\
3.16 I_N &= I_{SF}
\end{aligned}
$$

So the San Francisco earthquake was a little more than 3 times stronger than the Northridge earthquake.

Example 4:

The pH of milk is about 6.8. Find the hydrogen ion concentration.

Solution:

So for this example we will need the formula $pH = -\log\left(H^+\right)$. Notice that we are given the pH here as 6.8. Thus we simply need to insert it into the formula and solve for $H^+$.
We get

$$
\begin{aligned}
6.8 &= -\log\left(H^+\right) \\
-6.8 &= \log\left(H^+\right) \\
H^+ &= 10^{-6.8}
\end{aligned}
$$

So $H^+ = 10^{-6.8} \approx 0.000000158 \; moles/liter$ .

Example 5:

The intensity of a riveter at work is about 3.2 x $10^{-6}$ W/$m^2$. How loud in decibels is this sound level?

Solution:

For this example we need the formula $L = 10 \cdot \log \frac{I}{I_0}$ . Notice we are given $I$ = 3.2 x $10^{-6}$ W/$m^2$ and we already knew $I_0 = 10^{-12}$. So we simply need to plug the values in and solve. We get

$$
\begin{aligned}
L &= 10 \cdot \log \frac{3.2 \times 10^{-6}}{10^{-12}} \\
&= 10 \cdot \log\left(3.2 \times 10^{6}\right) \\
&= 65.05
\end{aligned}
$$

So the sound level is about 65 decibels.

Example 6:

A country in Africa had a population of 2 million in 2000 and 3 million in 2010. What would you predict the population of the country to be in 2020, assuming the growth rate is constant and exponential?

Solution:

First we need to recall the Exponential Growth and Decay Model, $y = Ce^{kt}$.

Now, we know the country's population in 2000 and 2010. Lets set these to be $t = 0$ and $t = 10$ respectively. This gives us the following

At $t = 0$, $y = 2\,million$
At $t = 10$, $y = 3\,million$

We want to know what happens at $t = 20$. But in order to do this we need to know what the values are of $C$ and $k$. So to do this we can insert the information that we know into our model. First we find $C$ as follows

$$y = Ce^{kt}$$
$$2 = Ce^{k\cdot 0}$$
$$2 = Ce^{0}$$
$$2 = C$$

Now our model is $y = 2e^{kt}$. We just need $k$. We use our other set of given information as follows

$$y = 2e^{kt}$$
$$3 = 2e^{10k}$$
$$\frac{3}{2} = e^{10k}$$
$$\ln\frac{3}{2} = 10k$$
$$0.0405 \approx k$$

Now our model is completed $y = 2e^{0.0405t}$. This model gives us the population (in millions) of the African country for any year we want after 2000. In particular we want 2020 or $t = 20$.

So we have
$$y = 2e^{0.0405t}$$
$$y = 2e^{0.040520}$$
$$y \approx 4.5\,million$$

Thus, the country in Africa will have 4.5 million people in 2020.

<u>Example 7</u>:

Radioactive iodine has a half-life of 60 days. If we start with 20 grams of radioactive iodine, how long will it take before we have 1 gram?

Solution:

First of all, half-life mean that after 60 days, half of the original amount is left.
This problem is an Exponential Decay Problem. So we again use the model $y = Ce^{kt}$.

Just like the last example we also need to find $C$ and $k$. This is done similar to the last example. First, we find $C$

$$\begin{aligned} y &= Ce^{kt} \\ 20 &= Ce^{k\cdot 0} \\ 20 &= C \end{aligned}$$

So our model is $y = 20e^{kt}$. Now, if we start with 20 grams and we let 60 days go by (60 days is the half-life) then we should have 10 grams left (half of 20).
We use this to find $k$.

$$\begin{aligned} y &= 20e^{kt} \\ 10 &= 20e^{60k} \\ \frac{1}{2} &= e^{60k} \\ \ln\frac{1}{2} &= 60k \\ -0.01155 &\approx k \end{aligned}$$

Now our model is complete $y = 20e^{-0.01155t}$. So we want to know the time required before we have 1 gram remaining. So we have

$$\begin{aligned} 1 &= 20e^{-0.01155t} \\ e^{-0.01155t} &= \frac{1}{20} \\ -0.01155t &= \ln\frac{1}{20} \\ t &\approx 259.4\, days \end{aligned}$$

So, the 20 grams will take 259.4 days to decay to 1 gram.

The applications using Newtons Law of Cooling are done in a similar fashion, i.e. first find the constant *k* and then use it to finish the problem.

The other applications we will see simply require us to interpret the formula given to determine the solution. Since this is something we have done many times before, we omit any example of this.

## 11.6 Exercises

1. Determine the balance after 20 years if $2000 is invested at 7% compounded (a) Daily and (b) continuously.

2. Determine the balance after 35 years if $10,000 is invested at 5.4% compounded (a) Monthly and (b) continuously.

3. Jon wants to invest some money. He wants to have $1,000,000. If Jon invests in an account that is compounded quarterly for 25 years at a rate of 9%, how much will Jon have to invest initially?

4. Jennifer has some extra money to invest. She wants to have an ending balance of $20,000 for a new car. If Jennifer invests in an account that is compounded daily for 7 years at a rate of 5%, how much will Jennifer have to invest initially?

5. A principle of $500 yields a balance of $1066.88 in 10 years when the interest is compounded continuously. What is the annual interest rate?

6. Matt invests a principle of $1500 which yields a balance of $3689.40 in 15 years. The interest is compounded continuously. What is the annual interest rate?

7. How much time is needed for $10,000 to double if it is invested in an account that compounds daily at a rate of 4.8%?

8. How much time is needed for $500 to double if it is invested in an account that compounds continuously at a rate of 6.4%?

9. Steve invests his life savings of $50 in a money market account. The bank offers him an interest rate of 3.8% and compounds his money bimonthly. How long will it take for Steve to have $500 in the account? How long will it take if the account compounds continuously?

10. Andrew wants to have $50,000 saved for college. He invests his $15,000 in a money market account. The bank offers him an interest rate of 6.1% and compounds his money daily. How long will it take for Andrew to save his desired amount? How long will it take if the account compounds continuously?

11. Jon is a little tight with his money. He only invests in very safe accounts. So Jon goes to the bank. The bank offers him two different accounts. The first account offers an interest rate of 5.3% and compounds it monthly. The second account offers an interest rate of 5.1% and compounds it continuously. Which account should Jon invest in if he wants his starting value of $2,000 to triple? How long will it take for his money to triple?

12. Tim invests $10,000 into a Certificate of deposit at Very Generous Bank. If the bank gives him a very generous 8.25% interest rate, and compounds it monthly, how much time is necessary for the money to triple? What about if the bank compounds it continuously?

13. On August 16, 1906, Chile had an earthquake that measured 8.6 on the Richer scale. What was the intensity of the earthquake in Chile?

14. In 1971, Los Angeles had an earthquake that measured 6.7 on the Richer scale. What was the intensity of the earthquake in Los Angeles?

15. On March 10, 1933, Long Beach, CA had an earthquake that measured 6.2 on the Richer scale. On December 3, 1988, an earthquake in Pasadena, CA measured 5.0 on the Richer scale. Compare the intensities of the two earthquakes. (Hint: See example 3)

16. In 1923, Tokyo, Japan had an earthquake that measured 8.3 on the Richer scale. In 1995, an earthquake in Kobe, Japan measured 7.2 on the Richer scale. Compare the intensities of the two earthquakes. (Hint: See example 3)

17. Find the pH of a solution that has a hydrogen ion concentration of $9.2\times10^{-8}$ moles per liter.

18. Find the pH of a solution that has a hydrogen ion concentration of $0.0000000054$ moles per liter.

19. The pH of water is 7. Find the hydrogen ion concentration of water.

20. The pH of a popular beverage is 4.5. Find the hydrogen ion concentration of the beverage.

21. A certain fruit has a pH of 2.5 and an antacid tablet has a pH of 9.5. The hydrogen ion concentration of the fruit is how many times the concentration of the tablet.

22. A certain soft drink has a pH of 6.5 and the competitor's drink has a pH of 5.9. How many times the hydrogen ion concentration of the competitors drink is that of the soft drink.

23. The intensity of a jackhammer is 4.6 x $10^{-5}$ W/$m^2$. How loud in decibels is this sound level?

24. The intensity of a car alarm is 6.2 x $10^{-9}$ W/$m^2$. How loud in decibels is this sound level?

25. At a recent rock concert the sound measured 120 dB. What was the intensity of the sound waves?

26. At Jon and Jennifer's cabin in Shaver Lake, the sound measured in the middle of the night is 12 dB. What is the intensity of the sound waves?

27. The intensity of the cheers when the Angels made the World Series was measured at 0.092 W/$m^2$. How loud in decibels is this sound level?

28. The intensity of the cheers at a San Francisco 49ers game is measured to be 0.00192 W/$m^2$. How loud in decibels is this sound level?

29. Jordan can yell at 54 dB. James can yell at 45 dB. How many more times intense is Jordan's yell than James' yell?

30. At a recent concert the band *Biohazard* had a measured sound of 138 dB. The warm up act *Infectious Waste* had a measured sound of 113 dB. How many more times intense is *Biohazard* 's music than *Infectious Waste*'s?

31. Radioactive radium has a half-life of 1620 years. If you start with 5 grams of isotope, how much remains after 1000 years?

32. The isotope $Pu^{230}$ has a half-life of 24,360 years. If you start with 10 grams of the isotope, how much remains after 10,000 years?

33. The isotope of $C^{14}$ has a half-life of 5730 years. If you start with a 15 gram sample, how long will it take until you have 10 grams left?

34. Radioactive radium has a half-life of 1620 years. If you start with 25 grams of isotope, how long will it take until you have only 5 grams left?

35. The isotope $Pu^{230}$ has a half-life of 24,360 years. If you start with 2 grams of the isotope, how long will it take to decay to a 0.85 gram sample?

36. The isotope $C^{14}$ has a half-life of 5730 years. If you start with 5 grams of this isotope, how much remains after 1,000,000 years?

37. The half-life of radioactive Iodine is 8 days. If we start with a 40 gram sample, what amount of radioactive iodine will remain after 100 days? How long will it take for the 40 gram sample to decay to a 5 gram sample?

38. The half-life of urea formaldehyde is 1 year. If we start with a 56 gram sample, what amount of urea formaldehyde will remain after 16 years? How long will it take for the 56 gram sample to decay to a 0.01 gram sample?

39. The isotope $C^{14}$ has a half-life of 5730 years. If after 1000 years you have 1.5 grams left, how much of the isotope did you start with?

40. The isotope $Pu^{230}$ has a half-life of 24,360 years. If after 100,000 years you have 23 grams left, how much of the isotope did you start with?

41. The half-life of radioactive Lead- 210 is 22 years. How long will it take until 30% of a sample of radioactive lead will remain?

42. The half-life of radioactive Iodine-131 is eight days. What percent of a present amount of radioactive iodine will remain after 23 days?

43. The half-life of radioactive Carbon-14, $\left({}^{14}C\right)$, is 5730 years. What percent of a present amount of radioactive carbon will remain after 100 years?

44. The isotope $Pu^{230}$ has a half-life of 24,360 years. What percent of a present amount of radioactive plutonium will remain after 5000 years?

45. The half-life of krypton-85 is 11 days. How long will it take for 6.3% of a given amount of krypton-85 to be left?

46. The half-life of Strontium-90 is 28 years. How long will it take for 1.4% of a given amount of Strontium-90 to be left?

47. In 2004, the city of Chicago had 6.5 million people. In 2012 they project Chicago will have 6.6 million people. If Chicago's population grows exponentially, how many people will live in Chicago in 2017?

48. Dhaka, Bangladesh had a population of 4.6 million people in 2012. Experts predict that in 2020 Dhaka, Bangladesh will have a population of 6.5 million. How many people will live in Dhaka, Bangladesh in 2025 if the growth rate remains constant and exponential?

49. The population in California in 1940 was 6.9 million people. In 1950 the population in California was 10.6 million people. If the population grows at a constant exponential rate, how many people will live in California in 2020?

50. In London, England, the population in 2010 was 9.17 million. The projected population in 2020 is 8.57 million. If the population in London is exponential, how many people will live in London in 2030?

51. In 2012, the Los Angeles had 10.1 million people. In 2020 they project Los Angeles will have 10.7 million people. If Los Angeles's population grows exponentially, how many people will live in Los Angeles in 2025? When will the population of Los Angeles reach 13 million people?

52. In Hanford, CA the population in 2000 was 21,000 and in 2010 was 30,800. If the population in Hanford is exponential, how many people will live in Hanford in 2030? In what year could we expect Hanford to have 100,000 people?

53. In Small Town USA the population in 2000 was 1,200 and in 2008 was 1,700. If the population in Small Town USA is exponential, how many people will live in Small Town USA in 2030? When will Small Town have a population of 10,000 people?

54. The population in Lagos, Nigeria in 2012 was 8.5 million people. In 2020 the experts project Lagos to have 12.5 million people. If the population grows at a constant exponential rate, when will Lagos have a population of 15 million people?

55. In Hometown USA the population in 2000 was 32,000 and in 2010 was 37,800. If the population in Hometown USA is exponential, how many people will live in Hometown USA in 2030? In what year will Hometown USA have 50,000 people?

56. The population in Visalia in 2000 was 75,000. In 2008, the population was 100,000. If the population of Visalia is growing at a constant exponential rate, how many people will live in Visalia in the year 2020? In what year will Visalia have 175,000 people?

57. A 2 grams culture of penicillin doubles in size in 1 hour. If penicillin grows exponentially, how long will it be before we have 10 grams of penicillin?

58. Happy Town USA had a population of 12,400 in 2003. In 2011 Happy Town USA had a population of 20,900. If the population of Happy Town USA grows at a constant exponential rate, when will Happy Town USA have a population of 30,000?

59. Your Town USA had a population of 150,400 in 2003. In 2011 Your Town USA had a population of 170,900. If the population of Your Town USA grows at a constant exponential rate, when will Your Town USA have a population of 200,000?

60. The theory of creation states in part that the human race started with only 2 people and that earth is around 5000 years old. Since the world population grows exponentially, we can verify if the theory of creation is consistent on these two points. The world population now is 6,500,000,000 people. Using a very conservative assumption of having every two people generate 0.5 of a person and taking 50 years to do so, is 5000 years enough time to get the earth's population to its current state?

Use Newtons Law of cooling described in this section for exercises 61-64 by first finding the constant value *k*.

61. A bottle of juice has a temperature of $73^\circ F$ and is left to cool in a refrigerator that has a temperature of $42^\circ F$. After 10 minutes, the temperature of the juice is $62^\circ F$. What will the temperature be after 17 minutes?

62. A bottle of soda has a temperature of $78^\circ F$ and is left to cool in a freezer that has a temperature of $25^\circ F$. After 5 minutes, the temperature of the soda is $70^\circ F$. What will the temperature be after 32 minutes?

63. A bottle of soda has a temperature of $78°F$ and is left to cool in a freezer that has a temperature of $25°F$. After 15 minutes, the temperature of the soda is $64°F$. When will the soda be a frosty $34°F$?

64. A glass of milk has a temperature of $70°F$ and is left to cool in a refrigerator that has a temperature of $45°F$. After 20 minutes, the temperature of the milk is $55°F$. When will the milk be a temperature of $50°F$?

65. The percentage P of market share for a product, *t* years after its introduction is given by

$$P = \frac{1}{1+e^{0.6t}}, \ t \geq 0.$$

How many years will it take before the market share drops to less than 10%?

66. After *x* months of training, the learning function $n = 800 - \frac{500}{e^{0.5x}}$ where *n* is the number of letters per hour that a postal clerk can sort. How many months of training are required for a clerk to sort 733 letters per hour?

67. College of the Sequoias is a campus of 10,000 students. One student returns from Thanksgiving weekend with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{10000}{1+9999e^{-0.8t}}, \ t \geq 0$$

where y is the total number infected after $t$ days. The administrators decide to cancel classes when 40% or more of the students are ill. How long will it take before the college will cancel classes?

68. The annual sales *y* of a product *x* years after it is introduced is modeled by $y = \frac{2000}{1+4e^{-x/2}}$.

At what time will the sales be 1100 units?

69. The time *t* in minutes for a small plane to climb to an altitude of *h* feet is given by $t = 50\log_{10}\frac{18000}{18000-h}$ where 18000 feet is the planes absolute ceiling. Find the time for the plane to climb to an altitude of 4000 feet. After 30 minutes, what is the altitude of the plane?

70. The demand for a certain product is given by $p = 500 - 0.5e^{0.004x}$ where *x* is the demand and *p* is the price. Find the demand when the price is $375.

71. The intensity of light entering water is reduced according to the model $I = I_0 e^{-kd}$, where $I$ is the intensity at a depth of $d$ feet, $I_0$ is the intensity at the surface, and $k$ is a constant called the *constant of extinction*. In Lake Kaweah, half the surface light remained at a depth of 26 feet. Find the depth at which 10% of the surface light remains.

72. Psychologists discovered that $f(t)=\dfrac{1}{1+e^{kt}}$ is an accurate model for describing the percent of correct responses after $t$ learning trials, where $k$ is a constant based upon a persons individual learning capabilities. If a certain person takes 10 learning trials to get 70% correct responses, how many trials will it take for the same person to respond over 90% correct?

73. Students in an intermediate algebra class like to forget everything they learn once they take the final. The average score they would get on the final exam *S,* given to them *t* months after the original final is given by $S(t)=65-15\log(t+1)$. What is the average score at the time of the final? What would the average score be 1 month after the final? When would the average score be 25?

74. The number of people who own a home in a new housing development is given by $N(t)=5+12\ln\left(\frac{t}{2}\right)$ where *N* is the number of homes that are owned and *t* is the time in months after the development opens. How many people owned a new home 2 months after opening? How many people owned a new home 2 years after opening? The development can only have 50 homes. How long will it take before all the homes have been bought?

75. The number of people who have heard a rumor increases exponentially. If every person who hears a rumor repeats it to two people a hour and if 5 people start the rumor then the number of people who have heard the rumor is given by $N(t)=5(3)^t$ where *t* is the time in hours. How many people will have heard the rumor in 4 hours? How long will it take for 1500 people to have heard the rumor?